تقدمان

دفتر

Calculus II

د. منى سكجها

إعداد الزميلة

ريم بركات

7.2 Volume :)

Volume

rotaition [revolving]

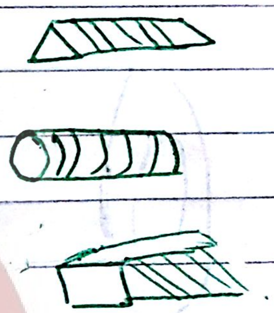
Washer — Disk

cylyindrical shell

Slicing

مساحة القاعدة

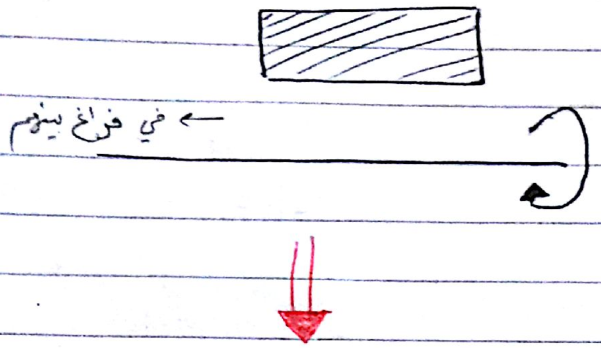
* semicride circle

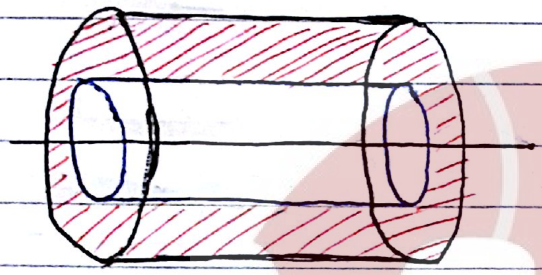


square Triangle → متطابق الضلعين → متطابق الأضلاع

Disk

* Washer :-

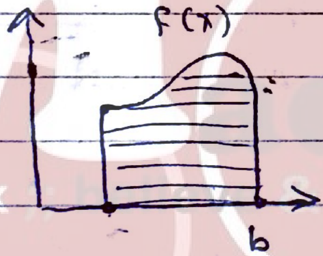




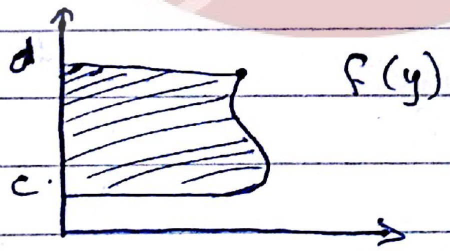
$$V = \underbrace{\pi r^2 h}_{\text{كلي}} - \underbrace{\pi r^2 h}_{\text{فراغ}}$$

$$= \pi\,(\underbrace{r^2}_{\text{كلي}} - \underbrace{r^2}_{\text{فراغ}})\,h$$

↳ Disk



$$V = \pi \int_a^b \left(\left(f(x)\right)\right)^2 \cdot dx$$



$$V = \pi \int_c^d \left(\left(f(y)\right)\right)^2 \cdot dy$$

* في فراغ ← نستعمل Washer

* ما في فراغ ← نستعمل Disk

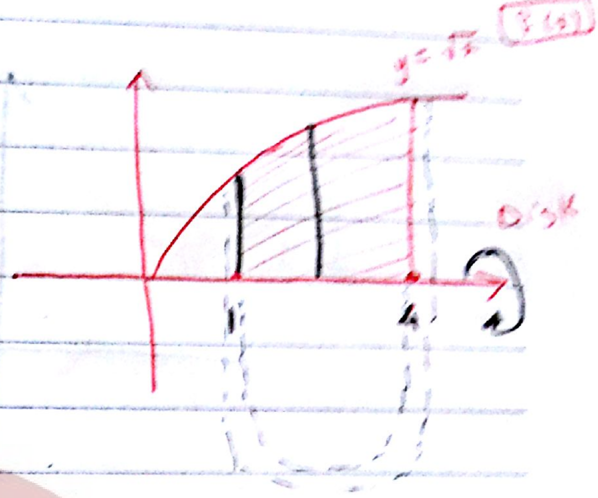
Ex:-

* Find the volume of the solid generated by rotating $y=\sqrt{x}$ on the interval $[1,4]$ about x-axis ?

$$V = \pi \int_1^4 (\sqrt{x})^2 \, dx$$

$$= \pi \left. \frac{x^2}{2} \right]_1^4$$

- - - - -

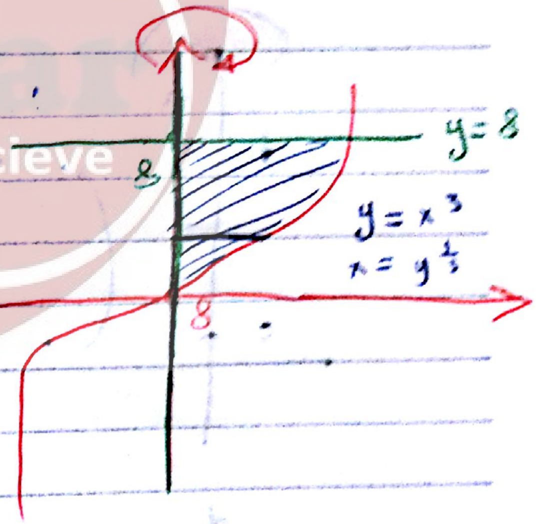


* Find the volume of the solid obtained by rotating the region enclosed by $y=x^3$, $y=8$, $x=0$ about y-axis ?

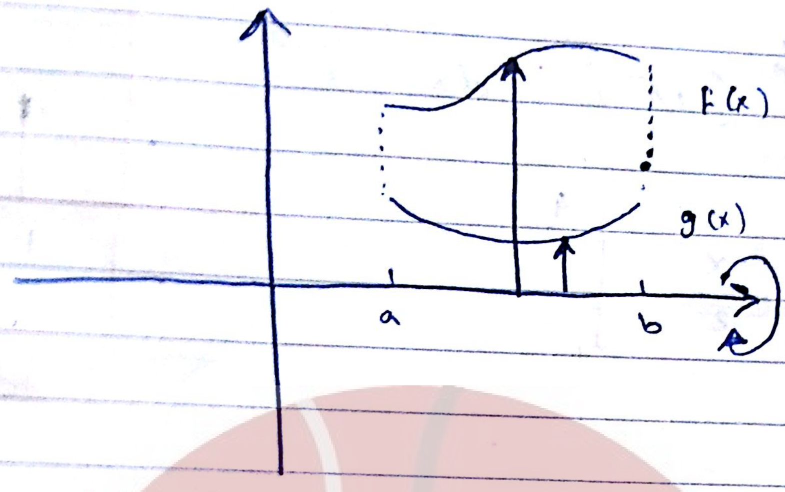
$$V = \pi \int_0^8 (y^{\frac{1}{3}})^2 \, dy$$

$$= \pi \int_0^8 y^{\frac{2}{3}} \, dy$$

$$= \left. \frac{\pi \, y^{\frac{5}{3}}}{5/3} \right]_0^8 \text{ - - - - -}$$

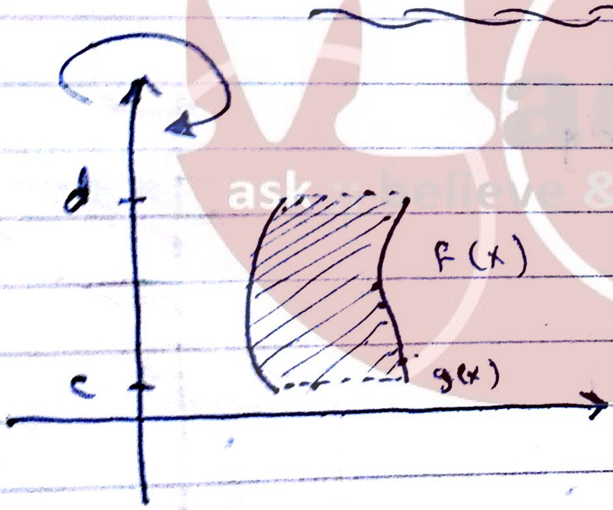


↳ Washer :



$$V = \pi \int_a^b \left((f(x))^2 - (g(x))^2 \right) . dx$$

نق كبير

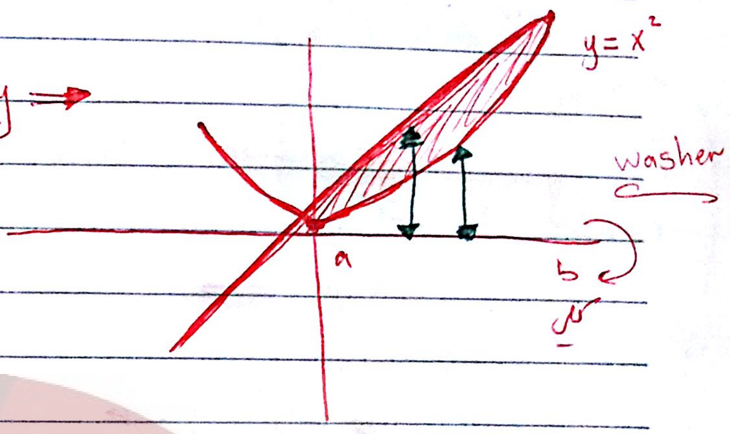


$$V = \pi \int_c^d (f(y))^2 - (g(y))^2 \; . dy$$

2013 / 2014

Ex:- Find the V of the region enclosed by $y = x$ and $y = x^2$, when rotating about x-axis ?

Sol:-



intersection pts $y = y \Rightarrow$

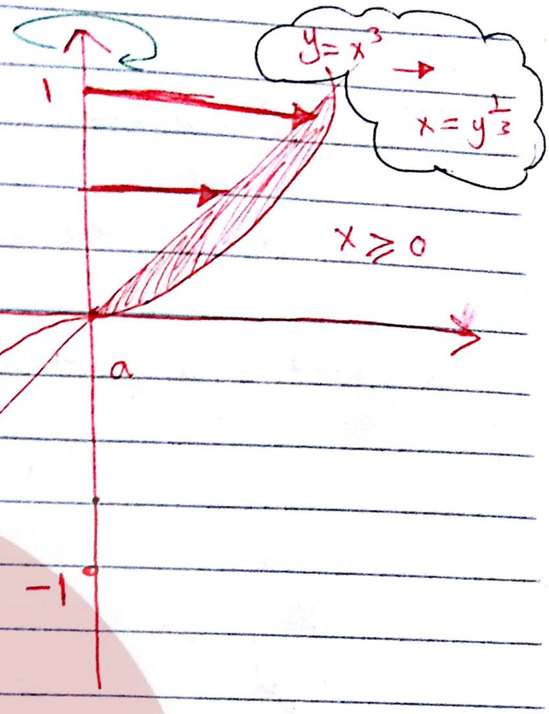
$x^2 - x \Rightarrow x^2 - x = 0$

$x(x-1) = 0$ — $x = 0 = a$, $x = 1 = b$

$$V = \pi \int_0^1 (x)^2 - (x^2)^2 \, dx$$

Ex:- Find the V of the region b by $y = x^2$, $y = x$, $x \geq 0$ when rotating about y-axis ?

Sol:-



intersection pts.

$x = x \Longrightarrow$

$y = y^{\frac{1}{3}} \Longrightarrow y^3 = y \Longrightarrow$

$y(y^2 - 1) = 0 \to y = 0,\ y = 1,\ y = -1$

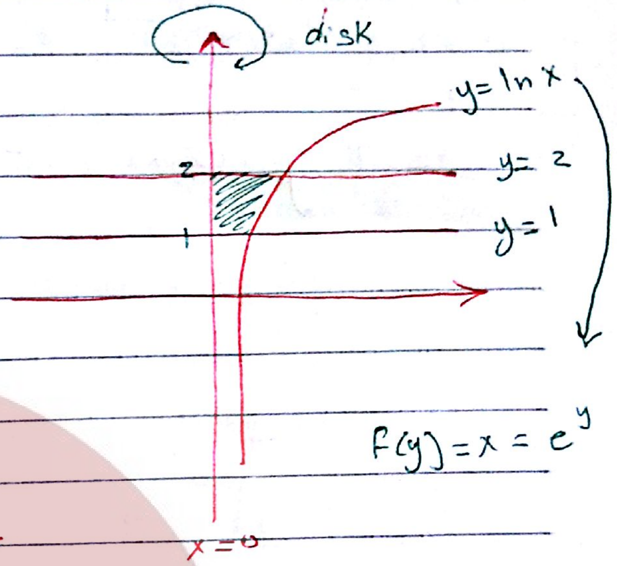
$$V = \pi \int_0^1 (y^{\frac{1}{3}})^2 - (y)^2 \cdot dy$$

$$= \pi \int_0^1 y^{\frac{2}{3}} - y^2 \, dy$$

$$= \pi \left[\frac{y^{\frac{5}{3}}}{5/3} - \frac{y^3}{3} \right]_0^1$$

Ex: Find the V of the region bdd by $y = \ln x$, $y = 1$, $y = 2$, $x = 0$ when revolving about y-axis?

Sol::



$$V = \pi \int_{1}^{2} (e^{y})^{2} \, dy$$

$$= \pi \int_{1}^{2} (e^{2y}) \, dy$$

$$= \pi \left. \frac{e^{2y}}{2} \right]_{1}^{2} \ldots\ldots$$

$f(y) = x = e^{y}$

يدور حولين x-axis ← كل الحل with respect to x

" " y-axis ← " " " y

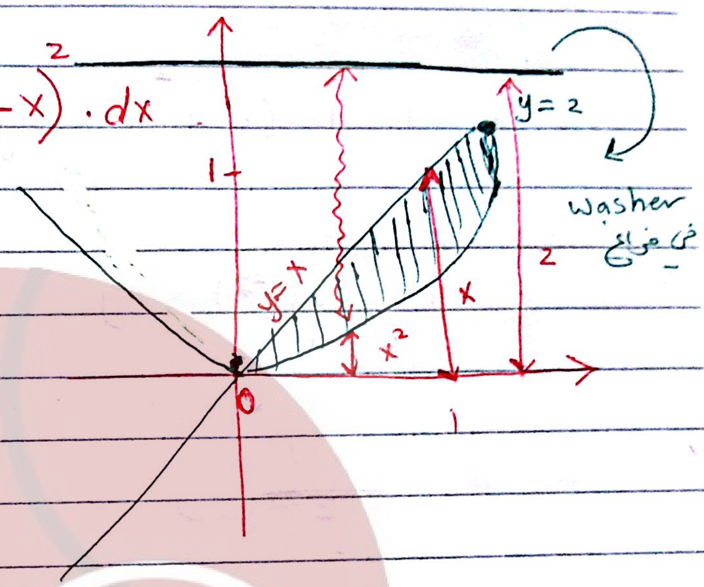
من تحت لفوق هي $f(x)$

هي $g(x)$ = =

Ex: Find V of the solid obtained by revolving the region enclosed by $y = x$, $y = x^2$ about $\boxed{y = 2}$ / about $\boxed{x = 2}$?

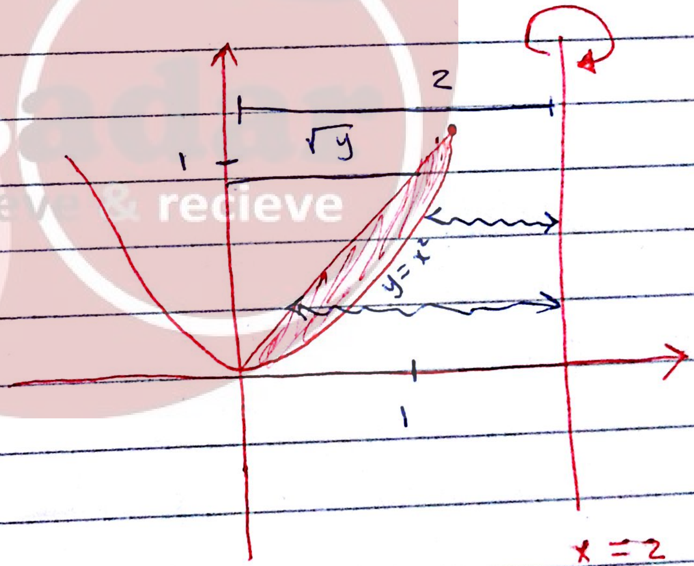
$\boxed{y = 2}$

$$V = \pi \int_0^1 (2 - x^2)^2 - (2 - x)^2 \cdot dx$$



$\boxed{x = 2}$

$$V = \pi \int_0^1 (2 - y)^2 - (2 - \sqrt{y})^2 \cdot dy$$



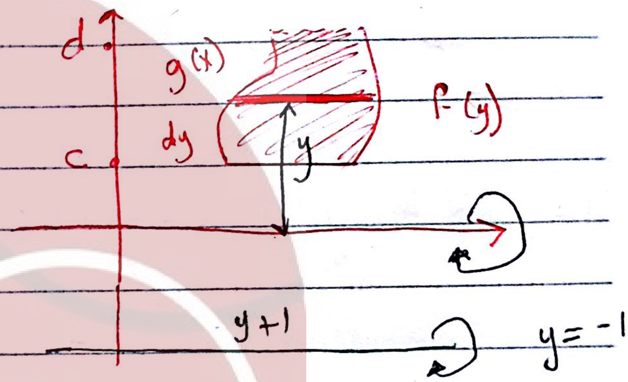
** 6.3 Cylindrial Shell :- C.S

Take the cross section // parallel to rotating axis.

[1] If rotating about x-axis.

$$V = 2\pi \int_{c}^{d} y \cdot \left(f(y) - g(y) \right) \cdot dy$$

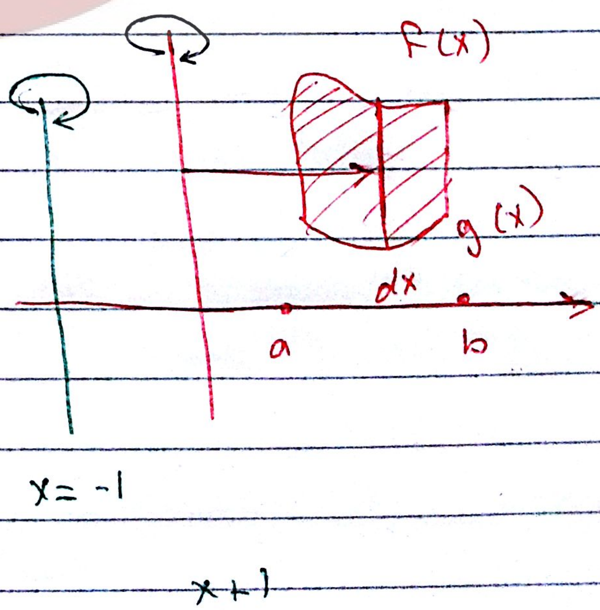
بعد المقطع عن محور السينات



[2] If rotating about y-axis.

$$V = 2\pi \int_{a}^{b} x \cdot \left(f(x) - g(x) \right) \cdot dx$$

بعد المقطع عن محور الصادات

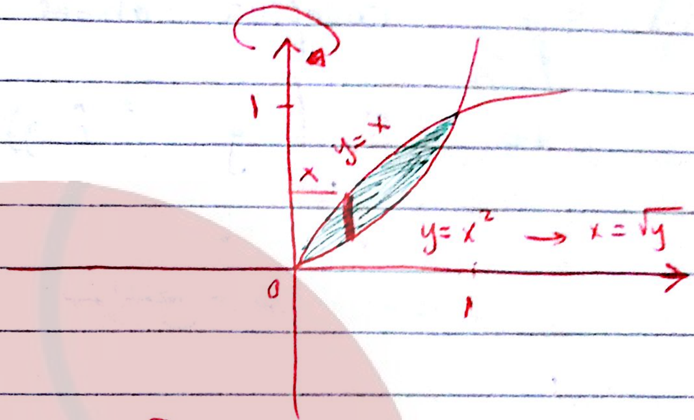


X حول محور الصادات Cross section دايما المعدين

y محور السينات ≠

* Ex:- Use C.S to find the V of the solid generated when the region in the first quadrant enclosed by $y=x$ and $y=x^2$ is revolving about y-axis?

Sol:-



intersection pts

$y=y \rightarrow x^2 = x$

$\Rightarrow x = 0, 1$

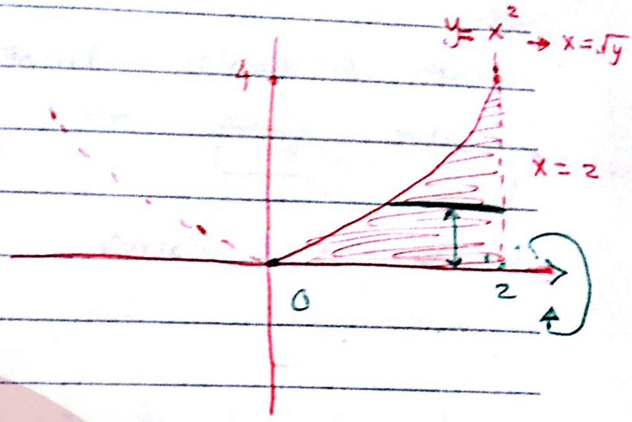
$$V = 2\pi \int_0^1 x\,(x - x^2)\,dx \quad \text{C.S}$$

Washer

$$V = \pi \int_0^1 (\sqrt{y})^2 - y^2 \; dy$$

* Ex: Use C.S to find the V of the Solid generated when the region under $y=x^2$ over $[0,2]$ is revolving about x-axis ?

Soli:-



C.S

$$V = 2\pi \int_0^4 y(2-\sqrt{y})\, dy$$

Disk

$$V = \pi \int_0^2 (x^2)^2 \cdot dx$$

Ex: Find the V of the solid generated when the region bdd by $y = 2x^2 - x^3$ and $y = 0$ in the first quadrant is revolving about y-axis ?

Sol:-

Since I can't find f(y) to solve using disk or washer

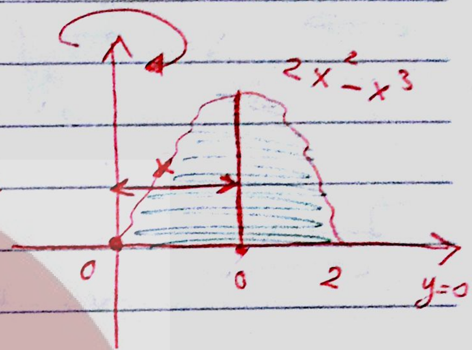
$2x^2 - x^3 = 0$

$x^2(2-x) = 0 \to x = 2, \ x = 0$

$\therefore$ Using C.S

$$V = 2\pi \int_0^2 x\,(2x^2 - x^3 - 0) \, dx$$

— — — —



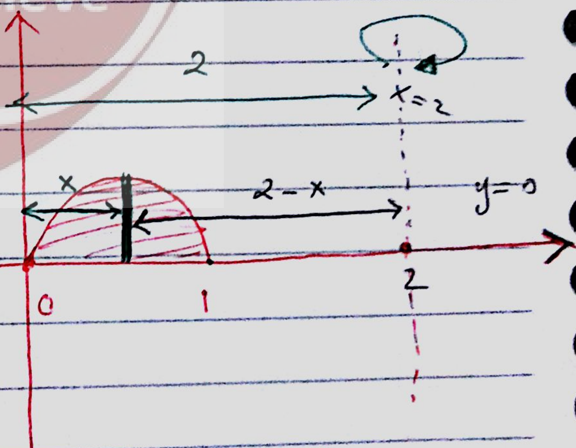
Ex:- Find V of the solid generated by rotating the region bdd. by $y = x - x^2$ and $y = 0$ about $x = 2$?

Sol:-

Using C.S

$$V = 2\pi \int_0^1 (2-x)\,(x - x^2 - 0)$$

— — — —



رسمتها عن طريق إكمال المربع

Methods of She

The volume of the solid whose base lie on the xy-plane and the cross section is parpendicular to x-axis is

$$\Rightarrow V = \int_a^b \underline{A(x)} \cdot dx$$

مساحة القاعدة بالنسبة لـ x

if the cross section is perpedicular to y-axis

$$\Rightarrow V = \int_c^d \underline{A(y)} \cdot dy$$

مساحة القاعدة بالنسبة لـ y

→ Area of square = $(\text{side})^2$

→ " " Semicircle = $\frac{1}{2} \pi r^2$

→ " " Triangle = $\frac{1}{2} * \text{base} * \text{height}$

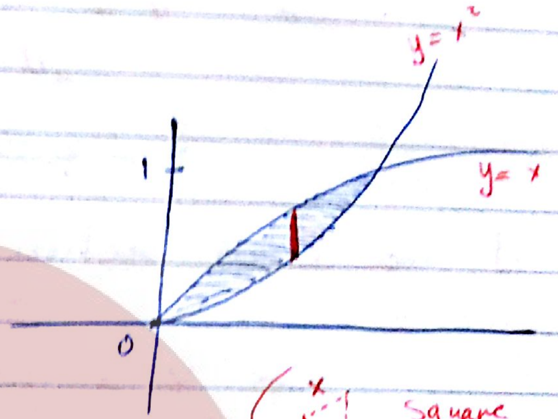
Remember **

Ex:- Find the V of the solid whose base enclosed by $y = x^2$ and $y = x$ and whose cross section is perpendicular to x-axis.

⇝ $\boxed{1}$ if the base is square.

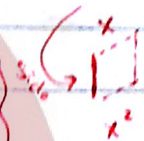
⇝ $\boxed{2}$ if the base is semicircle.

Sol:



$\boxed{1}$ $V = \int_a^b A(x)\,.\,dx$

$= \int_0^1 (x - x^2)^2 \,.\, dx$



side $= x - x^2$

$A(x) = (x - x^2)^2$

$\boxed{2}$ $2r = x - x^2 \rightarrow r = \dfrac{x - x^2}{2}$

$A(x) = \dfrac{1}{2}\pi r^2 = \dfrac{1}{2}\pi \left(\dfrac{x - x^2}{2}\right)^2$

$V = \int_0^1 \dfrac{1}{2}\pi \left(\dfrac{x - x^2}{2}\right)^2 . dx$

⇒ if ⊥ to y-axis

$y = \sqrt{y}$

Side $= (\sqrt{y} - y)$

→ $A(y) = (\sqrt{y} - y)^2$ ⇝ $V = \int_0^1 (\sqrt{y} - y)^2 . dy$

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★ Chapter 8 :

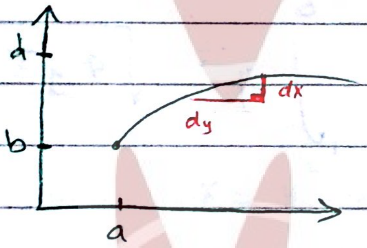
8.1 – Arc Length.

The arc length of the func. $f(x)$ from point (a,b) to point (c,d) is denoted by L,

x ← $$L = \int_a^c \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_a^c \sqrt{1 + f'(x)^2}\, dx$$

or

y ← $$L = \int_b^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_b^d \sqrt{1 + f'(y)^2}\, dy$$



★ **Ex.:** Find the arc length of the curve $y^2 = x^3$ between points $(1,1)$ to $(4,8)$?

W.r.t : $y^2 = x^3 \longrightarrow y = x^{\frac{3}{2}} = f(x)$

$$\frac{dy}{dx} = f'(x) = \frac{3}{2} x^{\frac{1}{2}} \longrightarrow \left(\frac{dy}{dx}\right)^2 = \frac{9}{4} x$$

$$L = \int_1^4 \sqrt{1 + \frac{9}{4} x}\; dx = \int_1^4 \left(1 + \frac{9}{4} x\right)^{\frac{1}{2}} dx$$

$$= \left. \frac{\left(1 + \frac{9}{4} x\right)^{\frac{3}{2}}}{\frac{3}{2}\left(\frac{9}{4}\right)} \right]_1^4 \ldots\ldots$$

Integration by Parts. $\boxed{1}$

Trig. $\boxed{2}$

* يلزم لطارة السلسلة من مادة الفيرست

تشاور

y

W.r.t.y $\quad y^2 = x^3 \longrightarrow x = y^{\frac{2}{3}} = f(y)$

$$\frac{dx}{dy} = \frac{2}{3} y^{-\frac{1}{3}} \longrightarrow \left(\frac{dx}{dy}\right)^2 = \frac{4}{9} y^{-\frac{2}{3}}$$

$$L = \int_1^8 \sqrt{1 + \frac{4}{9} y^{-\frac{2}{3}}} \, . dy = \int_1^8 \sqrt{1 + \frac{4}{9 y^{\frac{2}{3}}}} \, . dy$$

$$= \int_1^1 \sqrt{\frac{9 y^{\frac{2}{3}} + 4}{3 y^{\frac{1}{3}}}} \, . dy = \int_1^8 \frac{\sqrt{9 y^{\frac{2}{3}} + 4}}{\sqrt{9 y^{\frac{2}{3}}}} \, . dy$$

$$= \int_1^8 \frac{\sqrt{9 y^{\frac{2}{3}} + 4}}{3 y^{\frac{1}{3}}} \, . dy = \frac{1}{3} \int_1^8 y^{-\frac{1}{3}} \sqrt{9 y^{\frac{2}{3}} + 4} \, . dy$$

Let $\quad 9 y^{\frac{2}{3}} + 4 = z$

$$9 * \frac{2}{3} y^{-\frac{1}{3}} . dy = dz$$

$$6 \, y^{-\frac{1}{3}} . dy = dz \Longrightarrow y^{-\frac{1}{3}} dy = \frac{1}{6} dz$$

$$\frac{1}{3} * \frac{1}{6} \int \sqrt{z} \, . dz = \frac{1}{18} \frac{z^{\frac{3}{2}}}{\frac{3}{2}}$$

$$= \frac{1}{27} \left(9 y^{\frac{2}{3}} + 4\right)\Big|_1^8$$

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* Ex.: Set up the integral for the length of the curve of $x \cdot y = 1$ from $(1,1)$ to $(2, \frac{1}{2})$?

$$y = \frac{1}{x} \Longrightarrow \frac{dy}{dx} = \frac{1}{x^2} \Longrightarrow \left(\frac{dy}{dx}\right)^2 = \frac{1}{x^4}$$

$$L = \int_1^2 \sqrt{1 + \frac{1}{x^4}} \cdot dx$$

$$= \int_1^2 \sqrt{\frac{x^4 + 1}{x^4}}\, dx = \int_1^2 \frac{\sqrt{x^4+1}}{x^2}\, dx$$

توحيد مقامات

* Ex.: * Find the arc length for the curve of $x = y^2$ from $(0,0)$ to $(1,1)$?

w.r.t $\quad x = y^2 \longrightarrow \frac{dx}{dy} = 2y$

$$\left(\frac{dx}{dy}\right)^2 = 4y^2 \longrightarrow L = \int_0^1 \sqrt{1 + 4y^2} \cdot dy$$

Trig. Sub.

$$2y = 1 \tan\theta$$
$$4y^2 = \tan^2\theta$$
$$1 + 4y^2 = \underbrace{1 + \tan^2\theta}_{\sec^2\theta}$$
$$\sqrt{1 + 4y^2} = \sec\theta$$

$$2\,dy = \sec^2\theta \cdot d\theta \longrightarrow dy = \frac{1}{2}\sec^2\theta \cdot d\theta$$

$$L = \int \frac{1}{2} \underbrace{\sec\theta}_{u} \cdot \underbrace{\sec^2\theta}_{dv}$$

"By Parts"

** Area of the surface by rotation :-

The area of the surface obtained by rotating the curve of $y = f(x)$, $a \le x \le b$, $c \le y \le d$ about x-axis is given by :-

$$S = 2\pi \int_a^b f(x) \cdot \sqrt{1 + (f'(x))^2} \, dx$$

or

$$S = 2\pi \int_c^d y \cdot \sqrt{1 + (f'(y))^2} \cdot dy$$

$(2\pi y)$ محيط الدائرة

y : البعد بين محور السينات و curve

↳ if rotating about y-axis :-

$$S = 2\pi \int_c^d f(y) \cdot \sqrt{1 + (f'(y))^2} \, dy$$

or

$$S = 2\pi \int_a^b x \cdot \sqrt{1 + (f'(x))^2} \, dx$$

x : البعد بين محور الصادات و curve

PAGE
DATE

Ex: Find the surface area that is generated by rotating the curve of $y = x^3$, $0 \leq x \leq 1$ about x-axis ?

$$S = 2\pi \int_0^1 f(x) \cdot \sqrt{1 + (f'(x))^2} \cdot dx \quad \cancel{f(x) = x^3}$$

Sol:

$$f(x) = x^3 \longrightarrow f'(x) = 3x^2 \longrightarrow (f'(x))^2 = 9x^4$$

$$S = 2\pi \int_0^1 x^3 \cdot \sqrt{1 + 9x^4}$$

$$Z = 1 + 9x^4$$

- - - -

- - - - -

or

$$S = 2\pi \int_c^d y \cdot \sqrt{1 + (f'(y))^2} \, dy$$

$0 \leq x \leq 1 \longrightarrow x = 0 \longrightarrow y = x^3 = 0$, $x = 1 \longrightarrow y = 1$

$y = x^3 \longrightarrow x = y^{+\frac{1}{3}} = f(y) \longrightarrow (f'(y)) = \frac{1}{3} y^{-\frac{2}{3}}$

$(f'(y))^2 = \frac{1}{9} y^{-\frac{4}{3}}$, $S = 2\pi \int_0^1 y \cdot \sqrt{1 + \frac{1}{9} y^{-\frac{4}{3}}} \cdot dy$

$$= 2\pi \int_0^1 y \sqrt{1 + \frac{1}{9 y^{\frac{4}{3}}}} \cdot dy$$

Follow

Following

$$= 2\pi \int_0^1 y \sqrt{\frac{9y^{\frac{4}{3}}+1}{9y^{\frac{4}{3}}}} \cdot dy$$

$$= 2\pi \int_0^1 \frac{y}{3y^{\frac{2}{3}}} \sqrt{9y^{\frac{4}{3}}+1} \cdot dy$$

$$= 2\pi \int_0^1 \frac{1}{3} y^{\frac{1}{3}} \cdot \sqrt{9y^{\frac{4}{3}}+1} \cdot dy$$

$$Z = 9y^{\frac{4}{3}}+1$$

Ex: Find the surface area obtained by rotating $y = x^2$ $1 \leq x \leq 2$ about y-axis ?

Sol:

$$S = 2\pi \int_c^d f(y) \cdot \sqrt{1+(f'(y))^2} \cdot dy$$

$x = 1 \longrightarrow y = 1$, $x = 2 \longrightarrow y = 4$

$y = x^2 \longrightarrow x = \pm \sqrt{y} \Longrightarrow \boxed{x = f(y) = \sqrt{y}}$

$$f'(y) = \frac{1}{2\sqrt{y}}, \quad (f'(y))^2 = \frac{1}{4y}$$

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$$S = 2\pi \int_1^4 \sqrt{y} \cdot \sqrt{1 + \frac{1}{4y}} \cdot dy$$

$$= 2\pi \int_1^4 \sqrt{y} \cdot \sqrt{\frac{4y+1}{4y}} \cdot dy$$

$$= 2\pi \int_1^4 \sqrt{y} \cdot \frac{\sqrt{4y+1}}{2\sqrt{y}} \cdot dy$$

$$= \pi \int_1^4 (4y+1)^{\frac{1}{2}} \cdot dy = \pi \left. \frac{(4y+1)^{\frac{3}{2}}}{\frac{3}{2}(4)} \right]_1^4$$

or

$$S = 2\pi \int_1^2 x \cdot \sqrt{1 + (f'(x))^2} \cdot dx ,$$

$$f(x) = x^2 , \quad f'(x) = 2x .$$

$$(f'(x))^2 = 4x^2 \Longrightarrow S = 2\pi \int_1^2 x \cdot \sqrt{1 + 4x^2} \cdot dx$$

$$z = 1 + 4x^2$$

$$1 dz = 8x \, dx \quad \text{- - - - -}$$

*Ex.: Find the surface area that is generated by rotating the curve of $y = x + \ln x$, $2 \leq x \leq 3$ is rotating about y-axis ?

It's not easy to find $f(y)$ $\Rightarrow$ use the 2^{nd} rule

$$S = 2\pi \int_a^b x \cdot \sqrt{1 + (f'(x))^2} \cdot dx \Longrightarrow S = 2\pi \int_2^3 x \cdot \sqrt{1 + \left(1 + \frac{1}{x}\right)^2} \cdot dx \text{ - - - -}$$

* إذا كان دوران ال curve حول محور يوازي السينات
أو
محور يوازي الصادات
بستخدم البعد بيصير مثل طريقة ال Cylindrecal shell.

* انتبه انه حدود التكامل قد يكون منها تجزئة

Ex: Find the area of the surface obtained by revolving $y = \sqrt{4 - x^2}$, $-1 \leq x \leq 1$ about x - axis ?

Sol:

$$S = 2\pi \int_a^b f(x) \sqrt{1 + (f'(x))^2} \cdot dx$$

$$f(x) = (4 - x^2)^{\frac{1}{2}} \longrightarrow f'(x) = \frac{1}{\cancel{2}} (4 - x^2)^{-\frac{1}{2}} \cdot -\cancel{2}x$$

$$= \frac{-x}{\sqrt{4 - x^2}}$$

$$(f'(x))^2 = \frac{x^2}{4 - x^2}$$

$$S = 2\pi \int_{-1}^{1} \sqrt{4 - x^2} \cdot \sqrt{1 + \frac{x^2}{4 - x^2}} \cdot dx$$

$$= 2\pi \int_{-1}^{1} \sqrt{4 - x^2} \cdot \sqrt{\frac{4 - x^2 + x^2}{4 - x^2}} \cdot dx$$

$$= 4\pi \int_{-1}^{1} \sqrt{4-x^2} \cdot \frac{1}{\sqrt{4-x^2}}\, dx = 8\pi$$

Ex:- Find the surface area obtained by rotating $y = e^x$ $0 \le x \le 1$ about x-axis ?

Sol:

$$S = 2\pi \int_0^1 f(x) \cdot \sqrt{1 + (f'(x))^2} \cdot dx$$

$$f(x) = e^x \longrightarrow f'(x) = e^x.$$

$$(f'(x))^2 = (e^x)^2$$

$$= 2\pi \int_0^1 e^x \cdot \sqrt{1 + (e^x)^2} \cdot dx$$

$$z = e^x \longrightarrow 1\, dz = e^x \cdot dx$$

$$= 2\pi \int \sqrt{1+z^2} \cdot dz$$

$$z = \tan\theta$$

$$dz = \sec^2\theta \cdot d\theta$$

$$\sqrt{1+z^2} = \sqrt{1+\tan^2\theta} = \sqrt{\sec^2\theta} = \sec\theta$$

$$\int \sec\theta \cdot \sec^2\theta \cdot d\theta$$ By Parts.

3-4-2014

** Ch 11.

Sequance and Series :-

$a_1, a_2, a_3, a_4, a_5, \ldots\ldots, a_n, \ldots$ infinite sequence

First term, second term, n-th term

$$\left[a_n \right]_{n=1}^{\infty}$$ is the general form for the seq.

$a_1, a_3, a_5, a_7, \ldots$ odd terms

even terms.

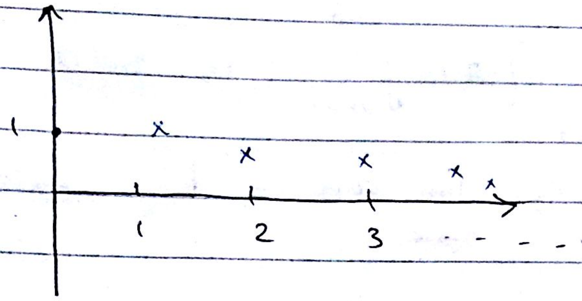
$a_2, a_4, a_6, \ldots$

Def: Sequences are functions with domain natural No. الاعداد

$$N = \{1, 2, 3, \ldots\}$$

So we can derive it, integrate it, take the limit for it, sketch it, ...

For example $\left\{ \frac{1}{n} \right\}_{n=1}^{\infty}$



Ex:- Find the general form for the seq.

① $\overset{2.1}{2}, \overset{2.2}{4}, \overset{3.3}{6}, 8, \dots$ $\qquad \{2n\}_{n=1}^{\infty}$

② $\overset{2^1}{2}, \overset{2^2}{4}, \overset{2^3}{8}, \dots$ $\qquad \{2^n\}_{n=1}^{\infty}$

③ $1, 3, 5, 7, \dots$ $\qquad \{2n-1\}_{n=1}^{\infty}$

④ $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5} \dots$ $\qquad \left\{ \frac{n}{n+1} \right\}_{n=1}^{\infty}$

⑤ $1, -1, 1, -1 \dots$ $\qquad \left\{ (-1)^{n+1} \right\}_{n=1}^{\infty}$

⑥ $\frac{1}{2}, \frac{-2}{3}, \frac{3}{4}, \frac{-4}{5}, \dots$ $\qquad \left[\frac{n}{n+1} (-1)^{n+1} \right]_{n=1}^{\infty}$

→ alternating sequences

* Remark :-

① The seq. $\{a_n\}_{n=1}^{\infty}$ is said to be convergent if

↳ $\lim_{n\to\infty} a_n = L$ exists

($L \neq \pm\infty$, $L \neq$ two numbers) otherwise it's divergent.

② The seq. $\{a_n\}_{n=1}^{\infty}$ is said to be convergent if both even terms and odd terms converge to ~~to~~ the same limit

ال odd , even
يروحوا لنفس ال limit

Ex:- Is the following conv. or div. ?

① $\left\{\frac{2n}{3n+1}\right\}_{n=7}^{\infty} = \lim_{n\to\infty} \frac{2n}{3n+1} = \frac{2}{3}$

conv.

② $\left\{\frac{n^2}{n+1}\right\}_{n=1}^{\infty} = \lim_{n\to\infty} \frac{n^2}{n+1} = \infty$ div.

$\lim\limits_{x \to \infty} \frac{\text{Poly}}{\text{Poly}}$

- درجة البسط > المقام ∞ , $-\infty$
- " البسط = المقام ~~[illegible]~~ $\frac{\text{معامل أكبر أس من البسط}}{\text{معامل أكبر أس من المقام}}$
- " البسط < المقام صفر

③ $\left\{ \frac{n+1}{n^2} \right\}_{n=5}^{\infty} = \lim\limits_{n \to \infty} \frac{n+1}{n^2} = 0$ conv.

④ $\left\{ 8 - 2n \right\}_{n=1}^{\infty} = \lim\limits_{n \to \infty} 8 - 2n = 8 - \infty = -\infty$ div.

⑤ $\left\{ \left(\frac{1}{2}\right)^n \right\}^{\infty} = \lim\limits_{n \to \infty} \left(\frac{1}{2}\right)^n = 0$ Conv.

$$\boxed{\begin{array}{ll} a^\infty = 0 & 0 < a < 1 \\ a^\infty = \infty & a > 1 \end{array}}$$

⑥ $\left\{ \left(\frac{3}{2}\right)^n \right\}_{n=2}^{\infty} = \lim\limits_{n \to \infty} \left(\frac{3}{2}\right)^n = \infty$ div.

⑦ $\left\{ \cos \frac{\pi}{n} \right\}_{n=1}^{\infty} = \lim\limits_{n \to \infty} \cos \frac{\pi}{n} = \cos \frac{\pi}{\infty}$

$= \cos 0 = 1$

conv.

⑧ $\left\{ \left(1 + \frac{2}{n}\right)^n \right\}_{n=1}^{\infty} = \lim\limits_{n \to \infty} \left(1 + \frac{2}{n}\right)^n = e^2$

conv.

$$\boxed{\lim_{n \to \infty} \left(1 + \frac{a}{n}\right)^n = e^a}$$

Alternating Series طالما تكون div. ال

و تبصير conv. إذا مرتبط ب sequence . يروح الصفر وهي غيرهيك div.

(9) $\{\cos n\}_{n=1}^{\infty} = \lim_{n\to\infty} \cos n = \cos \infty$ doesn't exit.

(10) $\left\{\frac{n}{e^n}\right\}_{n=1}^{\infty} = \lim_{n\to\infty} \frac{n}{e^n} \quad \frac{\infty}{\infty}!$ L'rule

$= \lim_{n\to\infty} \frac{1}{e^n} = \frac{1}{e^\infty} = \frac{1}{\infty} = 0$

(11) $\left\{\sqrt[n]{n}\right\}_{n=1}^{\infty} = \lim_{n\to\infty} \sqrt[n]{n} = \lim_{n\to\infty} n^{\frac{1}{n}} = \infty!$

$y = \sqrt[n]{n} \longrightarrow \ln y = \ln n^{\frac{1}{n}}$

$= \frac{1}{n} \ln n = \frac{\ln n}{n}$

$\lim_{n\to\infty} \ln y = \lim_{n\to\infty} \frac{\ln n}{n} \quad \frac{\infty}{\infty}!$ L'rule

$= \lim_{n\to\infty} \frac{\frac{1}{n}}{1} = 0$

$\lim y = \lim n^{\frac{1}{n}} = e^0 = 1$.

conv.

(12) $\left\{ \ln(n+1) - \ln n \right\}_{n=2}^{\infty}$

$\lim_{n\to\infty} \ln(n+1) - \ln n = \ln\infty - \ln\infty = \infty - \infty$

$\lim_{n\to\infty} \ln\left(\frac{n+1}{n}\right) = \ln\left(\lim_{n\to\infty} \frac{n+1}{n}\right) = \ln 1 = 0$

conv.

(13) $\left\{ (-1)^n \right\}^{\infty}$

$-1, 1, -1, 1, -1, \ldots\ldots$

odd term: $-1, -1, -1, \ldots \Rightarrow -1$

Even term: $1, 1, 1, 1 \ldots \Rightarrow 1$ doesn't exit

(14) $\left\{ \frac{1}{n} \cdot (1)^n \right\}_{n=1}^{\infty}$

$= \lim_{n\to\infty} \frac{1}{n} \cdot (-1)^n = 0 \cdot \pm 1 = 0$ conv.

(14) $\left\{ \frac{2n}{3n+1} \cdot (-1)^n \right\}_{n+1}^{\infty} = \lim \frac{2n}{3n+1} \cdot (-1)^n$

div $= \frac{2}{3}(\pm 1) = \pm \frac{2}{3}$

(15) $1, \frac{1}{2}, 1, \frac{1}{3}, 1, \frac{1}{4}, \ldots$

odd term $1, 1, 1, \ldots \Rightarrow 1$

$\neq$

Even term $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \Rightarrow 0$

div.

↳ Monotoric Sequences :- inc., dec.

$1, 2, 3, 4, \ldots$ increasing seq.

$f(n) = n$

$f'(n) = 1 > 0 \nearrow$

$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ decreasing seq.

$f(n) = \frac{1}{n}$

$f'(n) = \frac{-1}{n^2} < 0 \downarrow$

$1, 3, 5, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$

Errors

decreasing seq.

Recursive seq.:- "حدود يمكن إيجادها على بعضها"

$\sqrt{2}$, $\sqrt{2+\sqrt{2}}$, $\sqrt{2+\sqrt{2+\sqrt{2}}}$, ...

* Find the limit of seq. ?

$a_1 = \sqrt{2}$, $a_2 = \sqrt{2+a_1}$

$a_3 = \sqrt{2+a_2}$ ------ $\Longrightarrow$ $\boxed{a_{n+1} = \sqrt{2+a_n}}$

let $\lim\limits_{n\to\infty} a_n = L = \lim\limits_{n\to\infty} a_{n+1}$

$\lim\limits_{n\to\infty} a_{n+1} = \lim\limits_{n\to\infty} \sqrt{2+a_n} = \sqrt{2+\lim\limits_{n\to\infty} a_n}$

$L = \sqrt{2+L} \Longrightarrow L^2 = 2+L \Longrightarrow L^2 - L - 2 = 0 \Longrightarrow (L+1)(L-2) = 0$

$\boxed{\cancel{L=-1}}$ $\boxed{L=2}$

مستحيل $L=-1$ لسببين
الأول إنها متزايدة
والثانية هي الجذور مستحيل تروح لسالب

→ infinite series :-

given $a_1, a_2, a_3,$ ----- a_4, ----- infinite seq.

let $S_1 = a_1$ = first partial sum.

$S_2 = a_1 + a_2$ second partial sum.

$S_3 = a_1 + a_2 + a_3$ ضروري تنتهي بـ n بس مش ضروري تبدا بواحد

....

$S_n = a_1 + a_2 + \cdots\cdots a_n = \sum_{i=1}^{n} a_i$ = n-th partial sum.

** لو $\frac{1}{n}$ بتروح لقيمة : $\frac{1}{n-1}$, $\frac{1}{n+1}$ بتروح لنفس القيمة.

$$\lim_{n\to\infty} S_n = \sum_{i=1}^{\infty} a_i = \text{infinite series}$$

If $\lim_{n\to\infty} S_n = L$ exists $(\neq \pm\infty)$, the the infinite Series is convergent with sum $= L$, otherwise its divergent.

* Ex:- Is the following convergent or divergent.

[1] $\sum_{n=1}^{\infty} \tan^{-1}(n) = \tan^{-1}(n+1)$ a_n

Sol:-

$$S_n = \sum_{i=1}^{n} a_i = \sum_{i=1}^{n} \tan^{-1} i - \tan^{-1}(i+1)$$

$$S_n = (\tan^{-1} 1 - \cancel{\tan^{-1} 2}) + (\cancel{\tan^{-1} 2} - \cancel{\tan^{-1} 3}) +$$

$$(\cancel{\tan^{-1} 3} - \cancel{\tan^{-1} 4}) + \dots\dots +$$

$$(\cancel{\tan^{-1}(n-1)} - \cancel{\tan^{-1}(n)}) + (\cancel{\tan^{-1} n} - \tan^{-1}(n+1))$$

$\boxed{\text{الثاني يروح مع الأول بعده}}$

الفكرة

① اطلع Sn أولاً

② أفكها .

③ دايما أطلع الحد الأخير والي قبله (على الأقل حدين)

④ نعمل التشطيبات وهاي رح تحددلي

$$S_n = \tan^{-1} 1 - \tan^{-1}(n+1)$$

$$\lim_{n\to\infty} S_n = \tan^{-1} 1 - \tan^{-1}\infty$$

$$= \frac{\pi}{4} - \frac{\pi}{2} = -\frac{\pi}{4} \quad \text{Conv.}$$

$\equiv$ Sum of series.

(Telescoping series)

تأخذ مع أول الذي يليه

* ملحوظات

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$\boxed{2}$ $\sum_{n=2}^{\infty} \frac{1}{n^2+n}$ an

$$S_n = \sum_{i=2}^{n} a_i$$

$$= \sum_{i=2}^{n} \frac{1}{i} - \frac{1}{i+1}$$

$$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}$$

$$\frac{1}{n(n+1)} = \frac{A(n+1) + Bn}{n(n+1)}$$

$$1 = A(n+1) + Bn$$

let $n=0 \rightarrow \boxed{A=1}$

$n=-1 \rightarrow \boxed{B=-1}$

$$S_n = \left(\frac{1}{2} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{4}} - \frac{1}{3}\right) + \dots + \left(\cancel{\frac{1}{n-1}} - \cancel{\frac{1}{n}}\right) + \left(\cancel{\frac{1}{n}} - \frac{1}{n+1}\right)$$

تلغي مع أول يليه

$$S_n = \frac{1}{2} - \frac{1}{n-1}$$

$$\lim_{n\to\infty} S_n = \frac{1}{2} - \cancel{\frac{1}{\infty}}^{\,0} = \frac{1}{2} = \text{sum} \quad \text{Conv.}$$

(Telescoping)

H.W

تشطيب ال ∞ شو قبل تشطيب ال ٥ حدود

ع *

$$\sum_{n=1}^{\infty} \frac{1}{n^2+2n}$$

$n(n+2)$

ماخذ آخر ٣

ملاحظة: حد n ، $n-1$ ، $n-2$

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ملحوظة ⟵ الأقواس اللي فيها عدد زوجي من الحدود تكون علاقة التشطيب فيها أسهل من اللي فيها عدد فردي من الحدود .

$\boxed{3}$ $\displaystyle\sum_{n=2}^{\infty} \ln\left(1-\frac{1}{n^2}\right)$

ال n لازم أبدأ من 2
لأنه ln1 = صفر

$$\ln\left(1-\frac{1}{n^2}\right) = \ln\left(\frac{n^2-1}{n^2}\right) = \ln\left(\frac{(n-1)(n+1)}{n^2}\right)$$

$$= \ln(n-1) + \ln(n+1) - \ln n^2 = \ln(n+1) + \ln(n-1) - 2\ln n$$

$$= \ln(n+1) + \ln(n-1) - \ln n - \ln n$$

$$S_n = \sum_{i=2}^{n} \ln(i+1) + \ln(i-1) - \ln i - \ln i$$

حد أول

$$S_n = \left(\cancel{\ln 3} + \underline{\ln 1} - \cancel{\ln 2} - \underline{\ln 2}\right) + \left(\cancel{\ln 4} + \cancel{\ln 2} - \cancel{\ln 3} - \cancel{\ln 3}\right)$$

حد ثاني

$$+ \left(\cancel{\ln 5} + \cancel{\ln 3} - \cancel{\ln 4} - \cancel{\ln 4}\right) + \cdots\cdots$$

أول مع رابع يليه
ثالث مع ثاني

$$+ \left(\cancel{\ln(n)} + \cancel{\ln(n-2)} - \cancel{\ln(n-1)} - \cancel{\ln(n-1)}\right)$$

$$+ \left(\underline{\ln(n+1)} + \cancel{\ln(n-1)} - \underline{\ln n} - \cancel{\ln n}\right)$$

$$S_n = \ln 1 + \ln(n+1) - \ln n - \ln 2$$

$$\infty - \infty$$

$$\lim_{n\to\infty} S_n = -\ln 2 + \lim_{n\to\infty} \ln\frac{n+1}{n}$$

$$\ln 1 = 0$$

$$= -\ln 2$$

Power n للـ const. أي جيوميترك فكرتها

Tues.
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Gametric series

const.

$$a_1 + ar_2 + ar^2 + ar_n^3 \; - - - - - \; = \sum_{n=0}^{\infty} ar^n$$

$$\frac{ar}{n} = r \,,\; \frac{ar^2}{ar} = r \,,\; \frac{ar^3}{ar^2} = r \,,\; - - - \,,\; \boxed{r = base}$$

كل حد على الحد اللي قبله بيعطي r

Rule

$$\sum_{n=0}^{\infty} ar^n = \begin{cases} \text{conv. if } |r|<1 \text{ with sum} = \frac{a}{1-r} \\ \text{div. if } |r| \geq 1 \end{cases}$$

لازم يكون شرط انه n تبدأ من الصفر , (when n starts with n = 0)

* $\boxed{r}$ هي قيمة الثابت الذي يرفع للقوة n

* $\boxed{a}$:- هي الثابت المضروب بـ r^n عندما n تبدأ من الصفر

* لإيجاد a يجب معرفة أين بدأت n

* Ex:- Is the following Gamrator ~~div.~~ not

$\boxed{1}$ $\sum_{n=0}^{\infty} 2^n$

yes geometric

$\boxed{2}$ $\sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^{n+2}$, $\sum_{n=5}^{\infty} \left(\frac{-3}{4}\right)^{n+1}$

yes geometric

$\boxed{3}$ $\sum^{\infty} n^2$

not geometric

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$\boxed{4}$ $\sum_{n=1}^{\infty} n^n$

not Geometric

Ex: Is the following ~~Geometric or not~~ conv or div, if conv. Find the sum.

$\boxed{1}$ $\sum_{n=0}^{\infty} \left(\frac{3}{10}\right)^n$

$r = \frac{3}{10}$, $|r| = \left|\frac{3}{10}\right| = \frac{3}{10} < 1$

$\Rightarrow$ conv. $\Rightarrow a = 1 \Rightarrow$ sum $= \frac{a}{\;}$

دايما اطلع
ال abs. value ال
ال r

$\boxed{2}$ $\sum_{n=1}^{\infty} \left(\frac{-1}{2}\right)^n$

Note اول اشي ال
نعدل حدا ل الحال

$= \sum_{n=0} \left(\frac{-1}{2}\right)^{n+1}$

$= \sum_{n=0} \left(\frac{-1}{2}\right)\left(\frac{-1}{2}\right)^n$, $a = \frac{-1}{2}$, $r = \frac{-1}{2}$

$|r| = \frac{1}{2} < 1$ conv. sum $\frac{a}{1-r}$

$= \frac{-\frac{1}{2}}{1 - -\frac{1}{2}}$

③ $\sum_{n=0}^{\infty} \left(\frac{-3}{2}\right)^n$

$r = \frac{-3}{2}$, $|r| = \frac{3}{2} > 1$, div.

So ما نطلع الـ Sum

$r = \frac{1}{2}$, $\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n+1}$

$\frac{\frac{1}{4}}{\frac{1}{2}}$, ——— , ———

④ $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$

$r = \frac{1}{2} \rightarrow |r| = \frac{1}{2} < 1 \rightarrow$ conv. geometric

$\frac{\frac{1}{2}}{1} = \frac{1}{2}$

$\frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$

$\text{Sum} = \frac{a}{1-r} = \frac{1}{1-\frac{1}{2}} = 2$

* إذا كانت الـ series مفكوكة

وكان الحد الثاني على الأول = الثالث على الثاني = r

إذاً الحد الأول = a

* Note

$\sum_{n=1}^{\infty} a^n = a + a^2 + a^3 + \dots \equiv \sum_{n=1}^{\infty} a^{n+1} = a + a^2 + a^3 + \dots$

5) $\sum_{n=1}^{\infty} 4^{n-1}$

$= \sum_{n=0} 4^{(n+1)-1} = \sum_{n=0} 4^n$, $r = 4$,

$|r| = 4 > 1$ div. ما نطلع الـ sum.

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6) $\sum_{n=1}^{\infty} 2^{2n} \cdot 3^{1-n}$

$= \sum_{n=1} 4^n \cdot 3 \cdot 3^{-n}$

$\frac{4^n}{3^n} = \left(\frac{4}{3}\right)^n$

$= \sum_{n=1} 3\left(\frac{4}{3}\right)^n, \quad |r| = \left|\frac{4}{3}\right| > 1$ div.

7) $\sum_{n=1}^{\infty} \left(\frac{e}{\pi}\right)^n$

$= |r| = \frac{e}{\pi} = \left|\frac{2.7}{3.14}\right| < 1$ conv.

$\sum_{n=0} \left(\frac{e}{\pi}\right)^{n+1}$

$= \sum_{n=0} \left(\frac{e}{\pi}\right)\left(\frac{e}{\pi}\right)^n$

$\text{Sum} = \frac{a}{1-r} = \frac{e/\pi}{1 - e/\pi}$

imp. Note

$\sum_{n=1} a^{2n+1} = \sum_{n=0} a^{2(n+1)+1}$

لما تتأكد انه geometric معناها الحد الاول عبارة عن a

$\sum_{n=1} a^{2n} = \sum_{n=0} a^{2(n+1)} = \sum_{n=0} a^{2n+2}$

Ex: Write the following repeated decimal as ratio $\left(\dfrac{\text{whole } \#}{\text{whole } \#}\right)$ $0.3\overline{17}$?

~~Ex: 0.8~~, $0.3\overline{17} = 0.317171717\ldots$

$= 0.3 + 0.017 + 0.00017 + 0.0000017 + \ldots$

$\longrightarrow$ geometric series

$$\frac{0.00017}{0.017} = 10^{-2} = \frac{1}{100}$$

$$= \frac{0.0000017}{0.00017} = 10^{-2} = \frac{1}{100} = r$$

$$|r| = \frac{1}{100} < 1 \Rightarrow \text{conv.} \Rightarrow \text{Sum} = \frac{a}{1-r},$$

$$a = 0.017 \qquad \text{Sum} = \frac{0.017}{1 - 0.01} = \frac{0.017 * 10^3}{0.99 * 10^3} = \frac{17}{990}$$

$$= \frac{3}{10} + \frac{17}{990} = \boxed{\frac{3 * 99 + 17}{990}}$$

* To find the sum of the series (ملاحظة) there are only two ways

→ [1] $\lim S_n$ (لعند امتحان السكند بس هالطريقين)

→ [2] Geometric Series

*

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* To check wether the series is cov. or div.

1. $\lim S_n$
2. Geometric
3. divergent test

↳ Divergent test :-   * أهم إشي *

given $\sum_{n=()}^{\infty} a_n$

, If $\lim a_n \neq 0 \Longrightarrow \sum^{\infty} a_n$ is div.

If $\lim_{n\to\infty} = 0$ (test failed) (go to another test)

Ex: Is the following conv. or div.

1. $\sum_{n=2}^{\infty} \frac{2n^2+n}{3n^2+4n}$ an   ما بهتم من وين بتبلش

$\lim_{n\to\infty} \frac{2n^2+n}{3n^2+4n} \quad \frac{2}{3} \neq 0 \rightsquigarrow$ div.

ال S_n لازم على القابلية حدين .

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$\boxed{2}$ $\sum_{n=1}^{\infty} \left(\frac{3}{2}\right)^n$

$\lim_{n\to\infty} \left(\frac{3}{2}\right)^n = \infty \neq 0$ ← div.

or (using geo.)

$\boxed{3}$ $\sum_{n=1}^{\infty} \frac{1}{n}$

$\lim_{n\to\infty} \frac{1}{n} = 0$ (Test failed)

~~Notes~~:- * Notes:-

① $\sum (a_n \pm b_n) = \sum_n a_n + \sum_n b_n$

② $\sum_n c\, a_n = c \sum_n a_n$, c:- constant.

③ conv. + conv. $\longrightarrow$ conv.

conv. + div. $\longrightarrow$ div.

Ex: Find the sum :-

$\boxed{1}$ $\displaystyle\sum_{n=1}^{\infty} \frac{2^n + 3^n}{6^n}$

Sol:

$$\sum_{n=1} \frac{2^n}{6^n} + \sum_{n=1} \frac{3^n}{6^n}$$

$$= \sum_{n=1} \left(\frac{1}{3}\right)^n + \sum_{n=1} \left(\frac{1}{2}\right)^n$$

$|r| = \frac{1}{3} < 1$ $\qquad$ $|r| = \frac{1}{2} < 1$

geom. $\qquad$ gem.

$$= \sum_{n=1} \left(\frac{1}{3}\right)^{n+1} + \sum_{n=1} \left(\frac{1}{2}\right)^{n+1}$$

$= S_1 + S_2 =$ Sum.

② $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} + \left(\frac{1}{2}\right)^n$

$= \sum_{n=1} \frac{1}{n(n+1)} + \sum_{n=1} \left(\frac{1}{2}\right)^n$

$\frac{A}{n} + \frac{B}{n+1}$ geom

$\lim_{n\to\infty} S_n$ $\sum_{n=0}$

$S_1 + S_2 = \text{Sum}$

Sum:

① $\lim_{n\to a} S_n$

② geom $\frac{a}{1-r}$

Conv. or div.

① $\lim S_n$

② geom $|r|$

③ div. test

④ I.t

** 11.3 Integral test :- (I.T)

Given $\sum_{n=b}^{\infty} a_n$, if

[1] a_n has Positive terms (we can't use it with $(-1)^n$, cos n, Sin n

[2] a_n is continuos on $[b, \infty)$

[3] a_n is decreasing (i.e $a_n' \leq 0$)

$\longrightarrow$

then if

$$\int_b^\infty a(x) \cdot dx \text{ . conv.} \longleftrightarrow \sum_{n=b}^\infty a_n \text{ conv.}$$

$$\int_b^\infty a(x) \cdot dx \text{ . div} \longleftrightarrow \sum_{n=b}^\infty a_n \text{ . div.}$$

★ **Note:**

1. We use I.T with easy integration.
2. The Value of integral $\neq$ sum. of series.

Ex: Is the following conv. or div.

$\boxed{1}$ $\sum_{n=1}^{\infty} \frac{1}{n}$

$a_n = \frac{1}{n}$, $1, \frac{1}{2}, \frac{1}{3}, \ldots$ the terms.

$a_n = \frac{1}{n}$ (cont. on $[1,\infty)$ $\leftarrow$

$a_n' = -\frac{1}{n^2} < 0$ $\downarrow$

$\int_1^{\infty} \frac{1}{x}\,dx$. div $\Longrightarrow$ $\sum_{n=1}^{\infty} \frac{1}{n}$ div

$\boxed{2}$ $\sum_{n=1}^{\infty} \frac{1}{n^2}$

$a_n = \frac{1}{n^2}$ the terms

$a_n' = \frac{-2n}{n^2} < 0$ $\downarrow$ an cont.

$\int_1^{\infty} \frac{1}{x^2}\,dx$. conv. $\Longrightarrow$ $\sum_{n=1}^{\infty} \frac{1}{n^2}$

Rule: p- series $\sum_{p=1}^{\infty} \frac{1}{n^p}$ conv. if $p>1$

div. if $p \leq 1$

* Ex: $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{2}{3}}}$ div. p- series

$p = \frac{2}{3} < 1$

③ $\sum_{n=2}^{\infty} \frac{\ln n}{n}$

$a_n = \frac{\ln n}{n}$ +ve terms

a_n cont. on $[2, \infty)$

div. test

$\lim_{n \to \infty} \frac{\ln n}{n} = \frac{\infty}{\infty}$

L'rule $\lim_{n \to \infty} \frac{\frac{1}{n}}{1}$

$= 0$

$a_n' = \frac{1 - \ln n}{n^2} < 0$

when $1 - \ln n < 0$

$1 < \ln n$

$\int_2^{\infty} \frac{\ln x}{x} \, dx = \lim_{L \to \infty} \int_2^{L} \frac{\ln x}{x} \, dx$

$= \ln x = z \longrightarrow \frac{1}{x} dx = dz$

$$\int z \cdot dz = \frac{z^2}{2} = \left. \frac{(\ln x)^2}{2} \right]_2^L$$

$$\lim_{L \to \infty} \left(\frac{(\ln L)^2}{2} - \frac{(\ln 2)^2}{2} \right)$$

$$= \frac{(\ln \infty)^2}{2}$$

$= \infty -$ عدد $= \infty$ div.

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(4) $\sum_{n=1}^{\infty} \frac{e^n}{n}$

using div. test :-

$\lim_{n\to\infty} \frac{e^n}{n}$ $\frac{\infty}{\infty}$! L'rule

$\lim_{n\to\infty} \frac{e^n}{1} = \infty \neq 0$ div.

(5) $\sum_{n=1}^{\infty} n . e^{-n}$

$\lim_{n\to\infty} ne^{-n} = \cancel{\lim_{n\to\infty} \frac{n}{e^n}} \cancel{\frac{\infty}{\infty}}$! $\lim \frac{1}{e^n} = \frac{1}{\infty}$

$= 0$!

$a_n = \frac{n}{e^n}$ cont. , +ve terms ,

$a_n' = \frac{e^n . 1 - n . e^n}{e^{2n}}$

$= \frac{e^n (1-n)}{e^{2n}} < 0 \searrow$

Using I.T

$\int_1^{\infty} x . e^{-x} . dx = \lim_{L\to\infty} \int_1^{L} x . e^{-x} = x . \frac{e^{-x}}{-1}$

$+ \int e^{-x} . dx$

$= -x . e^{-x} - e^{-x} \Big]_1^L = \lim_{L\to\infty} \left(\left(-L . e^{-L} - e^{-L} \right) - \left(-e^{-1} - e^{-1} \right) \right)$

$-\infty . 0$!! $\frac{\cancel{e}}{\infty}$

عدد

$\lim \frac{-L}{e^L} = 0 = $ Conv.

* Ex:- Is the following conv. or div.

[1] $\sum_{1}^{\infty} \frac{1}{n^2+1}$ conv.

Using Integral test :-

+ ve terms , cont. on $[1, \infty)$

$f'n = \frac{-2n}{(n^2+1)} < 0$ ↓

$\int_1^\infty \frac{1}{x^2+1} dx = \lim_{L \to \infty} \int_1^L \frac{1}{x^2+1} dx =$

$\tan^{-1} x \Big]_1^L = \tan^{-1} L - \tan^{-1} 1 = \tan^{-1} \infty - \tan^{-1} 1$

$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$ conv.

[2] $\sum_{1}^{\infty} \frac{n}{n^2+1}$

Using I.T : +ve terms , cont.

$f_n' < 0$ ↓

$\int_1^\infty \frac{x}{x^2+1} dx = \lim_{L \to \infty} \int_1^L \frac{x}{1+x^2} \cdot dx$

$= \lim_{L \to \infty} \frac{1}{2} \ln |1+x^2| \Big]_1^L$

$= \lim_{L \to \infty} \left(\frac{1}{2} \ln |1+L^2| - \frac{1}{2} \underbrace{\ln |1+1|}_{2} \right)$

$= \frac{1}{2} \ln \infty = \infty$ div.

** 11.4 Comparison test, limit comparison test :-

Comparison Test :- C.T

$\sum a_n$, $\sum b_n$ if a_n , b_n have positive terms

s.t $a_n \le b_n$, then

$\boxed{1}$ if $\sum b_n$ conv. $\longrightarrow$ $\sum a_n$ conv.
$\boxed{2}$ if $\sum a_n$ div. $\longrightarrow$ $\sum b_n$ div.

المهدف
اما تكون كبيرة
conv.
أو صغيرة
div.

(Use it with easy fractions).

Ex: (Is the following fractions conv. or div.)

$\boxed{1}$ $\displaystyle\sum_{1}^{\infty} \frac{1}{\sqrt[3]{n} - \frac{1}{2}}$ +ve terms.

$$\sqrt[3]{n} - \frac{1}{2} < \sqrt[3]{n} = n^{\frac{1}{3}}$$

$$\frac{1}{\sqrt[3]{n} - \frac{1}{2}} > \frac{1}{n^{\frac{1}{3}}}$$

نعني الصغيرة

$\displaystyle\sum_{1}^{\infty} \frac{1}{n^{\frac{1}{3}}}$ $\quad P = \frac{1}{3} < 1$ div.

C.T

" Comparison Test "

** 11.4 Comparison test, limit comparison test :-

Comparison Test :- C.T

$\sum a_n$, $\sum b_n$ if a_n , b_n have positive terms

s.t $a_n \leq b_n$, then

$\boxed{1}$ if $\sum b_n$ conv. $\longrightarrow$ $\sum a_n$ conv.

$\boxed{2}$ if $\sum a_n$ div. $\longrightarrow$ $\sum b_n$ div.

الهدف
اما تكون كبيرة
conv.
أو صغيرة
div.

(Use it with easy fractions) .

Ex: (Is the following fractions conv. or div.)

$\boxed{1}$ $\sum_{1}^{\infty} \frac{1}{\sqrt[3]{n} - \frac{1}{2}}$ +ve terms.

$$\sqrt[3]{n} - \frac{1}{2} < \sqrt[3]{n} = n^{\frac{1}{3}}$$

$$\frac{1}{\sqrt[3]{n} - \frac{1}{2}} > \frac{1}{n^{\frac{1}{3}}}$$

نمشي الصغيرة

$$\sum_{1}^{\infty} \frac{1}{n^{\frac{1}{3}}} \qquad P = \frac{1}{3} < 1 \qquad \text{div.}$$

C.T
" Comparison Test "

② $\sum_{1}^{\infty} \frac{1}{3n^2+n}$

$3n^2+n > 0$

lim S_n نرجع على مرحلة الاختصار

$\frac{1}{3n^2+n} < \frac{1}{n}$ الكبير div.

∴ ما بستفيد

$3n^2+n > n^2$

$\frac{1}{3n^2+n} < \frac{1}{n^2}$

$\sum_{1}^{\infty} \frac{1}{n^2}$ $p=2>1$

Conv. Conv.

C.T

③ $\sum_{1}^{\infty} \frac{1}{n-\frac{1}{2}}$

you can use I.T also

$n-\frac{1}{2} < n$

+ve terms

$\frac{1}{n-\frac{1}{2}} > \frac{1}{n}$

$\sum_{1}^{\infty} \frac{1}{n}$ $p=1$

div. div.

C.T

④ $\sum_{1}^{\infty} \frac{1}{3n^2+5n+7}$

$3n^2+5n+7 > n^2$

$\frac{1}{3n^2+5n+7} < \frac{1}{n^2}$ C.T

Conv. $\sum \frac{1}{n^2}$ Conv. $p=2>1$

⑤ $\sum \frac{1}{2^n+1}$

$2^n+1 > 2^n$

$\frac{1}{2^n+1} < \frac{1}{2^n} = \left(\frac{1}{2}\right)^n$

$\sum \left(\frac{1}{2}\right)^n$ geom

Conv. $\boxed{C.T}$ $|r| = \frac{1}{2} < 1$

Conv.

⑥ $\sum \frac{1}{2^n-1}$

$2^n-1 < 2^n$

$\frac{1}{2^n-1} > \frac{1}{2^n} = \left(\frac{1}{2}\right)^n$

$\sum \left(\frac{1}{2}\right)^n$ Conv. Test fails

لا bn حط أكبر أسس البسط على أكبر أسس بالمقام

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Limit Comparison Test. (L.C.T)

Given $\sum a_n$, make $\sum b_n$ s.t a_n, b_n are of the +ve terms.

Take $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = L$

اللي انا حطيته

[1] If $L > 0$ and $\sum b_n$ conv. $\rightarrow$ $\sum a_n$ conv.
and $\sum b_n$ div. $\rightarrow$ $\sum a_n$ div.

[2] If $L = 0$ and $\sum b_n$ conv. $\rightarrow$ $\sum a_n$ conv.

[3] If $L = \infty$ and $\sum b_n$ div. $\rightarrow$ $\sum a_n$ div.

Use it usually with fractions كسور

Ex: Is the following conv. or div.

① $\sum\limits_{1}^{\infty} \left(\frac{3n^2 + 5}{7n^5 + 3n + 1} \right)$ (a_n) L.C.T

$b_n = \frac{n^2}{n^5} = \frac{1}{n^3}$, $\lim\limits_{n\to\infty} \frac{a_n}{b_n}$

$\lim\limits_{n\to\infty} \frac{3n^2 + 5}{7n^5 + 3n + 1} * \frac{n^3}{1} = \frac{3}{7}$ معامل / معامل

$\sum b_n = \sum \frac{1}{n^3}$ p-series $p = 3 > 1$ conv.

$\Rightarrow \sum a_n$ conv.

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(2) $\sum_{1}^{\infty} \frac{1}{2^n - 1}$ an

$b_n = \frac{1}{2^n}$, $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{2^n - 1} * \frac{2^n}{1} = 1 > 0$

$\sum b_n = \sum \frac{1}{2^n} = \sum \left(\frac{1}{2}\right)^n$

geom. conv.

$|r| < 1 \Rightarrow \sum a_n$ conv

(3) $\sum_{1}^{\infty} \frac{1}{\sqrt[3]{n} + 5}$

$\sqrt[3]{n} + 5 > \sqrt[3]{n}$ c.T

$\frac{1}{\sqrt[3]{n} + 5} < \frac{1}{\sqrt[3]{n}} = \frac{1}{n^{\frac{1}{3}}}$

div.

Using L.C.T $b_n = \frac{1}{\sqrt[3]{n}} \rightarrow$

$\Rightarrow \lim_{n\to\infty} \frac{1}{\sqrt[3]{n} + 5} * \frac{\sqrt[3]{n}}{1} = 1 > 0$

$\sum b_n = \sum \frac{1}{n^{\frac{1}{3}}}$ $p = \frac{1}{3} < 1$ div. $\Rightarrow \sum a_n$ div.

(4) $\sum_{1}^{\infty} \frac{1}{n - \frac{1}{4}}$ an

*a) using I.T it's div.

*b) c.T $= n - \frac{1}{4} < n$

$\frac{1}{n - \frac{1}{4}} > \frac{1}{n}$

div

*c) L.C.T $b_n = \frac{1}{n}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{n - \frac{1}{4}} * \frac{n}{1} = 1 > 0$

$\sum b_n = \sum \frac{1}{n}$ div. $\rightarrow \sum a_n$ div.

5) $\displaystyle\sum_{1}^{\infty} \frac{3n^2 + 3n}{\sqrt{1 + 2n^5}}$

$$b_n = \frac{n^2}{n^{\frac{5}{2}}} = \frac{1}{n^{\frac{1}{2}}}$$

$$\lim_{n \to \infty} \frac{3n^2 + 3n}{\sqrt{1 + 2n^5}} \times \frac{n^{\frac{1}{2}}}{1} = \frac{3}{\sqrt{2}} > 0 ,$$

$$\sum b_n = \sum \frac{1}{n^{\frac{1}{2}}} \quad \text{"div"}$$

$\therefore$ an div.

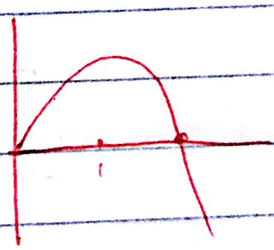
(*) Rule **

- $\displaystyle\lim_{n \to 0} \frac{\sin n}{n} = 1$

- $\displaystyle\lim_{n \to \infty} \frac{\sin\left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)} = 1$

- $\displaystyle\sum_{1}^{\infty} \sin\left(\frac{1}{n}\right)$

  +ve دائماً



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6) $\sum_{1}^{\infty} \sin\left(\frac{1}{n}\right)$ an

+ve terms

$b_n = \frac{1}{n} \rightarrow \lim_{n\to\infty} \frac{a_n}{b_n}$

$= \lim \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = 1 > 0$

$\sum b_n = \sum \frac{1}{n}$ div. $p = 1$

$\rightsquigarrow \sum a_n$ div.

7) $\sum_{1}^{\infty} \sin\left(\frac{1}{n^2}\right)$ an

+ve terms

$b_n = \frac{1}{n^2}$, $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sin\left(\frac{1}{n^2}\right)}{\frac{1}{n^2}} = 1 > 0$

$\sum b_n = \sum \frac{1}{n^2}$ conv. $p = 2$

$\sum a_n$ conv.

8) $\sum_{1}^{\infty} \left(1 + \frac{1}{n}\right)^2 \cdot e^{-n}$

$= \sum \frac{\left(1+\frac{1}{n}\right)^2}{e^n}$

$b_n = \frac{1}{e^n}$, $\lim \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)^2}{\cancel{e^n}} * \frac{\cancel{e^n}}{1}$

$= 1 > 0$

$\sum b_n = \sum \frac{1}{e^n} = \sum \left(\frac{1}{e}\right)^n$ geom

$|r| \cong \frac{1}{2.7} < 1$ Conv.

$\Rightarrow \sum a_n$ Conv.

Q:- If $S_n = \frac{n-1}{n+1}$, Find $\sum_{n=1}^{\infty} a_n$, a_n ?

$$S_n = \sum_{i=1}^{n} a_i \Longrightarrow \lim_{n\to\infty} S_n = \sum_{i=1}^{\infty} a_i$$

$$= \sum_{n=1}^{\infty} a_n$$

$$= \lim_{n\to\infty} \frac{n-1}{n+1} = 1$$

$$S_n = \underbrace{a_1 + a_2 + \cdots\cdots + a_{n-1}} + a_n$$

$$S_n = S_{n-1} + a_n$$

$$S_n - S_{n-1} = a_n$$

$$a_n = \frac{n-1}{n+1} - \frac{(n-1)-1}{(n-1)+1}$$

$$= \frac{n-1}{n+1} - \frac{n-2}{n} = \frac{(n-1)n - (n-2)(n+1)}{n(n+1)}$$

* Is the following conv. or div :-

① $\sum_{2}^{\infty} \frac{\ln n}{n}$ a_n

(using I.T it's div.)

مالها I.T

* using L.C.T , $b_n = \frac{1}{n}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\ln n}{n} * \frac{n}{1} = \ln\infty = \underline{\underline{\infty}}$$

$\sum \frac{1}{n}$ div.

$\Longrightarrow \sum a_n$ div.

② $\sum_{1}^{\infty} \frac{\ln n}{\sqrt{n}}$ a_n

using L.C.T $b_n = \frac{1}{\sqrt{n}} = \frac{1}{n^{\frac{1}{2}}}$

ما بنقدر نستخدم I.T هون صعبة

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\ln n}{\cancel{\sqrt{n}}} * \frac{\cancel{\sqrt{n}}}{1} = \lim n = \infty$$

$\sum b_n = \sum \frac{1}{n^{\frac{1}{2}}}$ div. $p = \frac{1}{2} < 1$

$\Longrightarrow \sum a_n$ div.

$\underline{\underline{\text{H.W}}}$ $\sum_{2}^{\infty} \frac{\ln n}{n^2}$ / $b_n = \frac{1}{n^{1.5}}$ $\frac{1}{n^{1.5}}$

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** 11.6. Ratio and Root test :-

Ratio test :- use it with series with +ve terms
(use it usually with factorial and power of
n , $e.n^{()}$, $(n)^{(n)}$, $n^{()}$ مضروب

~~total~~ * take $\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = L$

أول ما تشوف ال factorial بتخدم ratio

دائما ال factorial موجب

(a) if $L < 1 \Longrightarrow \sum a_n$ conv.
(b) if $L > 1$ or $L = \infty \Longrightarrow \sum a_n$ div
(c) if $L = 1$ (test fails) use another test.

If a_n has no +ve terms and you want to use ratio you can take $\lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} = L$ and continue.

* Ex:-

Is the following conv. or div.

① $\sum_{n=1}^{\infty} \frac{1}{n!}$

$a_n = \frac{1}{(n+1)!}$

$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{1}{(n+1)!} * \frac{n!}{1}$

$\lim_{n \to \infty} \frac{n!}{(n+1)(n)!} = \frac{1}{\infty} = 0 < 0$ case 1

$\sum a_n$ conv. Ratio test

$0! = 1$, $1! = 1$, $3! = 3.2.1 = 3.2!$

* $\boxed{n! = n(n-1)(n-2) \cdots\cdots 3.2.1}$

$n(n-1)!$

② $\sum_{1}^{\infty} \frac{n^n}{2^n}$ a_n

$a_n = \frac{(n+1)}{2^{(n+1)}}$

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{\lim (n+1)^{n+1}}{2^{n+1}} = \frac{2^n}{n^n}$

$\lim_{n\to\infty} \frac{(n+1)^n \cdot (n+1)}{\cancel{2^n} \cdot 2} * \frac{\cancel{2^n}}{n^n}$

$\lim_{n\to\infty} \frac{n+1}{2} * \frac{(n+1)^n}{n^n}$

$\infty = \lim \left(\frac{(n+1)}{n}\right)^n$

$\infty = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$

$\infty . e^{-1} = \infty$ div.

$\Rightarrow \sum a_n$ div.

-1^n و 1^n فرق بين

مرة سالب مرة +

③ $\sum_{1}^{\infty} \frac{(n!)^2}{(2n)!}$ a_n

$a_n = \frac{((n+1)!)^2}{(2(n+1)!)} = \frac{(n+1)! * (n+1)!}{(2n+2)!}$

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} =$

$\lim \frac{(n+1)! \cdot (n+1)!}{(2n+1)!} * \frac{(2n)!}{n! \cdot n!}$

$\lim_{n\to\infty} \frac{(n+1)\cancel{n!} \cdot (n+1)n!}{(2n+1)(2n+1)\cancel{(2n!)}} * \frac{\cancel{(2n)!}}{\cancel{n! \cdot n!}}$

$\lim \frac{(n+1)\cdot(n+1)}{(2n+1)(2n+1)} = \frac{1}{4} < 1$

$\sum a_n$ conv.

* Is the following conv. or div.

$\boxed{*}$ $\{1^n\}_{n=2}^{\infty}$

$\lim_{n\to\infty} 1^n = 1$

$1, 1, 1, 1, \ldots \Longrightarrow 1$

conv.

* Is the series conv. or div. :-

$$\sum_{n=2}^{\infty} 1^n$$

geom $|r| = 1 \Longrightarrow$ div.

→ Root test :-

Given $\sum a_n$, a_n +ve terms

take $\lim_{n\to\infty} (a_n)^{\frac{1}{n}} = \lim_{n\to\infty} \sqrt[n]{a_n} = L$

(a) If $L < 1 \Longrightarrow \sum a_n$ conv.
(b) If $L > 1$ or $\infty \Longrightarrow \sum a_n$ div.
(c) If $L = 1$ (test fails)

Use it usually with powers of n,
if n has -ve terms you can make
$\lim_{n\to\infty} (|a_n|)^{\frac{1}{n}} = L$ .....
continue.

Ex.:

① $\sum_{1}^{\infty} \left(\frac{4n-5}{2n+1}\right)^n$ a_n

$\lim_{n\to\infty} (a_n)^{\frac{1}{n}} = \lim_{n\to\infty} \left(\left(\frac{4n-5}{2n+1}\right)^{\cancel{n}}\right)^{\cancel{\frac{1}{n}}} = \frac{4}{2} = 2 > 1$

$\sum a_n$ div.

② $\sum_{1}^{\infty} \frac{1}{(\ln(n+1))^n}$

$\lim_{n\to\infty} \left(\frac{1}{(\ln(n+1))^n}\right)^{\frac{1}{n}} = \lim_{n\to\infty} \frac{1^{\frac{1}{n}}}{((\ln(n+1))^{\cancel{n}})^{\cancel{\frac{1}{n}}}}$

$= \lim_{n\to\infty} \frac{1}{\ln(n+1)} = \frac{1}{\ln\infty} = \frac{1}{\infty} = 0 < 1$

$\sum a_n$ conv.

$$③ \quad \sum_{1}^{\infty} n\left(\frac{1}{2}\right)^{n} \quad a_n$$

(we can use ratio)

$$= \lim_{n\to\infty} \left(n\left(\frac{1}{2}\right)^{n}\right)^{\frac{1}{n}} = \frac{1}{2} \lim_{n\to\infty} n^{\frac{1}{n}}$$

$\frac{1}{2}$

∞^{0} !!

$\Downarrow$

$(1) * \frac{1}{2} = \frac{1}{2} < 1$

$\therefore \sum a_n$ Conv.

$$y = n^{\frac{1}{n}} \Longrightarrow \ln y = \ln n^{\frac{1}{n}} = \frac{1}{n} \ln n$$

$$\lim \ln y = \lim_{n\to\infty} \frac{\ln n}{n} = \frac{\infty}{\infty}\ !$$

L'rule

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

$$e^{0} = 1$$

* نهاية مادة الفيرست *

Sun.
20 - 4 - 2014

** Alternating Test :- (متذبذبة)

Alternating series is of form

$$\sum_{1}^{\infty} (-1)^n a_n = -a_1 + a_2 - a_3 + a_4 \ldots\ldots$$

or

$$\sum_{1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + \ldots$$

Alternating Test :- Given $\sum (-1)^{n} a_n$ ($n-1$, $n+1$)

if ① a_n is decreasing seq. (i.e تغير ذو $a_n' < 0$)

and

② $\lim_{n\to\infty} a_n = 0$

إذا تحقق هذين الشرطين ⇐ مباشرة ∴ conv.

Then $\sum (-1)^n a_n$ is converget.

* note :-

when you see alternating series

First check ① divergence test.
② Alternating test
③ Rotating Root
④ Convergence test.

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Ex: Is the following conv. or div.

$\boxed{1}$ $\sum (-1)^n \frac{2n+1}{3n+1}$ a_n

using div. test.

$\lim_{n\to\infty} (-1)^n \cdot \frac{2n+1}{3n+1} = \pm 1 \cdot \frac{2}{3} = \pm \frac{2}{3} \neq 0$

div.

$\boxed{2}$ $\sum (-1)^n \cdot \frac{1}{n}$

using alternating test $a_n = \frac{1}{n}$,

$a_n' = \frac{-1}{n^2} < 0$ ↓

$\lim_{n\to\infty} \frac{1}{n} = 0 \Longrightarrow$ Conv.

$\boxed{3}$ $\sum_{3}^{\infty} (-1)^n \cdot \frac{\ln n}{n}$

using alternating test $a_n = \frac{\ln n}{n}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\ln n}{n} = 0$

$a_n' = \frac{n \cdot \frac{1}{n} - \ln n \cdot 1}{n^2} < 0$ ↓

$\sum (-1)^n \frac{\ln n}{n}$ Conv.

ملحوظة مهمة

$\sum \frac{1}{n}$ div.

$\sum (-1)^n \cdot \frac{1}{n}$ conv.

④ $\sum (-1)^n \dfrac{n}{\ln n}$

divergence test

$$\lim_{n\to\infty} (-1)^n \cdot \frac{n}{\ln n}$$

$$\pm 1 . \lim_{n\to\infty} \frac{n}{\ln n}$$ L'rule $\dfrac{1}{\frac{1}{n}} = n$

$$\pm 1 . \infty = \pm \infty \neq 0$$

$\therefore$ div.

** Convergence test :

If $\sum |a_n|$ conv. $\Longrightarrow$ $\sum a_n$ conv.

$\sum a_n$

① $\frac{1}{2} - \frac{1}{2^2} - \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} \cdots\cdots$

Sol.

taking $|\ |$ $\Longrightarrow$ $\sum |a_n|$ $\Longrightarrow$ $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots$

$= \sum_{n=1} \left(\frac{1}{2}\right)^n$ geom. $|r| = \frac{1}{2} < 1$ conv.

using conv. test $\sum a_n$ conv.

② $\sum_{1}^{\infty} \frac{\cos n}{n^2}$

using convergence test.

take $| \ |$ $\Rightarrow$ $\sum \left| \frac{\cos n}{n^2} \right| = \sum \frac{|\cos n|}{n^2}$

$|\cos n| \leq 1 \Rightarrow \frac{|\cos n|}{n^2} \leq \frac{1}{n^2}$

conv. $p = 2 > 1$

$\sum \frac{1}{n^2}$ conv. using C.T $\Rightarrow \sum \frac{|\cos n|}{n^2}$ conv.

using Convergence test

$\sum \frac{\cos n}{n^2} \Rightarrow$ conv.

③ $\sum_{1}^{\infty} \frac{\sin(3n)}{n^4+1}$

using convergence test, take $| \ |$

$\sum \left| \frac{\sin 3n}{n^4+1} \right| = \sum \frac{|\sin 3n|}{n^4+1}$

$n^4+1 > n^4$

$\frac{1}{n^4+1} < \frac{1}{n^4}$

$\frac{|\sin 3n|}{n^4+1} \leq \frac{1}{n^4+1} < \frac{1}{n^4}$

conv. conv. conv.

$p = 4 > 1$

following using Comparison Test

$\sum \frac{|\sin 3n|}{n^4+1}$ conv. $\xrightarrow[\text{Convergence Test}]{\text{using}}$ $\sum \frac{\sin(3n)}{n^4+1}$

is conv.

Q.: Is the following conv. or div.

$\boxed{1}$. $\sum_{1}^{\infty} \ln\left(\frac{n}{4n+1}\right)$

using divergence test $\lim_{n\to\infty} \ln \frac{n}{4n+1}$

$= \ln\left(\lim_{n\to\infty} \frac{n}{4n+1}\right)$

$= \ln \frac{1}{4} \neq 0$ div.

$\boxed{2}$ $\sum (-1)^n \cos\left(\frac{\pi}{n}\right)$

using div. test $\lim_{n\to\infty} (-1)^n \cos \frac{\pi}{n} =$

$= \pm 1 . \cos \frac{\pi}{\infty}$

$= \pm 1 . \cos 0$

$\pm 1 . 1 = \pm 1 \neq 0$

div.

*Q:- Is the following

$$\sum_{1}^{\infty} (-1) \frac{1}{n!} \quad \text{Conv. or div.}$$

$$|a_n| = \left| (-1)^n \cdot \frac{1}{n!} \right| = \frac{1}{n!}$$

$$|a_{n+1}| = \frac{1}{(n+1)!}$$

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{1}{(n+1)!} * \frac{n!}{1}$$

$$\lim_{n\to\infty} \frac{1}{(n+1)\cdot n!} * \frac{n!}{1} = \frac{1}{\infty} = 0 < 1$$

$\Rightarrow$ Conv.

**Def:

① A series $\sum a_n$ is said to be Absolute Convergent if $\sum |a_n|$ is convergent

② A series $\sum a_n$ is said to be conditionally convergent if ① $\sum |a_n|$ is div. and $\sum a_n$ is conv. ②

*Q: Which of the following is Absolute convergent or Conditionally convergent.

* [1] $\sum_{1}^{\infty} (-1)^n \left(\frac{1}{2}\right)^n$

taking $| \, |$ $\Rightarrow \left(\frac{1}{2}\right)^n$ geom $|r| = \frac{1}{2} < 1 \rightarrow$ conv.

↳ Absolutely Convergent.

* [2] $\sum_{1}^{\infty} (-1)^n \cdot \frac{1}{n}$

~~tat~~ taking $| \, |$, $\sum \frac{1}{n}$ div $p=1$, but

$\sum (-1)^n \cdot \frac{1}{n}$ conv. by Alternating test.

↳ Conditionally Convergent.

* [3] $\sum (-1)^n \cdot \frac{1}{n^2}$

taking $| \, |$, $\sum \frac{1}{n^2}$ conv. $P = 2 > 1$

↳ Absolutely Convergent

** 11.8 Power of series :-

$$\sum_{n=0}^{\infty} C_n x^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \cdots\cdots$$

is power series in powers of x.

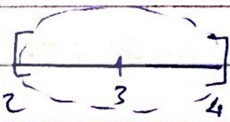
$$\sum_{n=0}^{\infty} C_n (x-a)^n = C_0 + C_1 (x-a) + C_2 (x-a)^2 + \cdots\cdots$$

is power series in powers of $(x-a)$

* Radius of the interval :-

① $[2,4]$

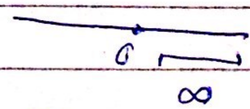
$$\text{rad.} = \frac{4-2}{2} = 1$$



② $(2,4)$

$$\text{rad.} = \frac{4-2}{2} = 1$$

③ $(-\infty, \infty)$



$$\frac{\infty - -\infty}{2} = \frac{\infty}{2} = \infty$$

④ $\{a\}$

$$\text{rad.} = \frac{a-a}{2} = 0$$

↳ Interval of convergent :-

these values of x that make the series convergent.

To find Interval of convergence use either Ratio or root test

* Find the interval of convergence & the radius.

① $\sum_{1}^{\infty} x^n a_n$

using root test

* سهلة إنك تحلها *
بال ratio
وبال root

$$\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = \lim_{n\to\infty} |x^n|^{\frac{1}{n}}$$

$$= |x| < 1$$

بال root
و ratio

عشان تطلع conv.
لازم أخليها أقل من ١

$|x| > 1 \implies -1 \leq x \leq 1$

At $-1 \implies \sum (-1)^n$ geom $|r| = |-1| = 1$ div.

At $1 \implies \sum 1^n$ geom $|r| = 1$ div.

Interval of conv. is $(-1, 1)$

Rad $= 1$

a_n

[2] $\sum_{1}^{\infty} \frac{(x-3)^n}{n}$ صعبة root في هاي المسألة بال

عشان تصير $n^{\frac{1}{n}}$

Using Ratio test $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$

$|a_n| = \frac{|x-3|^n}{n}$, $|a_{n+1}| = \frac{|x-3|^{n+1}}{n+1}$

$\lim_{n\to\infty} \frac{(x-3)^{n+1}}{n+1} * \frac{n}{|x-3|^n}$

$\lim_{n\to\infty} \frac{|x-3| \cdot \cancel{|x-3|^n}}{n+1} \cdot \frac{n}{\cancel{|x-3|^n}}$

$= |x-3| \cdot \lim_{n\to\infty} \frac{n}{n+1}$

$= |x-3| * 1 = |x-3| < 1$

$-1 < x-3 < 1 \Longrightarrow 2 < x < 4$

At $2 \longrightarrow \sum \frac{(2-3)^n}{n} = \sum \frac{(-1)^n}{n}$

Conv. By Alternating test.

③ $\sum_{1}^{\infty} \overset{a_n}{n!\, x^n}$

$$|a_n| = n!\, |x|^n$$

$$|a_{n+1}| = (n+1)!\,.\,|x|^{n+1}$$

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{(n+1)!\,|x|^{n+1}}{n!\,|x|^n}$$

$$= \lim_{n\to\infty} \frac{(n+1)\,.\,\cancel{a!}\,.\,\cancel{|x|^n}\,.\,|x|}{\cancel{a!}\,.\,\cancel{|x|^n}} = |x|\,.\lim_{n\to\infty}(n+1)$$

$$= \infty$$

$\forall\, x$ except $x = 0$

★ ايش يعني ∞ ؟؟

لجميع قيم x ماعدا الصفر تكون div.

At $x = 0 \implies \sum n! * 0$

$$= \sum 0 = 0 + 0 + 0 + \cdots$$

$$= 0$$

whenever of convergence

$\{0\}$

④ $\sum_{1}^{\infty} \frac{(-1)^n \cdot x^{2n}}{2^{2n} (n!)^2}$ a_n

$|a_n| = \frac{|x|^{2n}}{2^n \cdot n! \cdot n!}$, $|a_n| = \frac{|x|^{2n+2}}{2^{2n+2} * n! * n!}$

$\lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} = \lim \frac{\cancel{|x|^{2n}} \cdot |x|^2}{\cancel{2^{2n}} * 2^2 * (n+1) \cancel{n!} \cdot (n+1) \cancel{n!}} *$

$\frac{\cancel{2^{2n}} \; n! \cdot n!}{|x|^{2n}}$

$\lim_{n \to \infty} \frac{|x|^2}{4(n+1)^2} = 0 < 1$

interval of convergence $= \mathbb{R} = (-\infty, \infty)$

$R_{cd} = \infty$

** Taylor and Maclauin Series :-

Def :-

① $\sum_{\boxed{n=0}}^{\infty} f^{(n)} \frac{(a)(x-a)^n}{a!} = f(a) + \frac{f'(a)(x-a)}{1!} +$

$f(a) + \frac{f''(a)(x-a)^2}{2!} + \ldots$

Taylor series = Power series in Powers of $(x-a)$

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② $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)\, x^n}{n!} = f(a) + \frac{f'(a)\, x}{1!} + \frac{f''(a)\, x^2}{2!} + \cdots$

~~Mach~~ Maculains Series = Taylor series when $a=0$, Power series in Powers of x

Ex.: Find the Maculains Series for the following.

① $f(x) = e^x$

$f(0) + \frac{f'(0)\, x}{1!} + \frac{f''(0)\, x^2}{2!} + \cdots$

$1 + \frac{1 \cdot x}{1!} + \frac{1 \cdot x^2}{2!} + \frac{1 \cdot x^3}{3!} + \cdots$

$f(x) = e^x \longrightarrow f(0) = 1$

$f'(x) = e^x \longrightarrow f'(0) = 1$

$f''(x) = e^x \quad\quad 1$

$\vdots \quad\quad \vdots$

$= \sum_{n=0} \frac{x^n}{n!} = e^x$

② $f(x) = e^{-x}$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \longrightarrow e^{-x} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!}$$

$$\text{Yep} \quad e^{-x} = \sum_{n=0} \frac{(-1)^n \cdot x^n}{n!}$$

③ $f(x) = \sinh x$

$$= \frac{e^x - e^{-x}}{2}$$

$$e^x = \sum_{n=0} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

$$e^{-x} = \sum_{n=0} \frac{(-1)^n x^n}{n!} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \cdots$$

$$e^x - e^{-x} = \frac{2x}{1!} + \frac{2x^3}{3!} + \frac{2x^5}{3!} + \cdots$$

$$\frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

$$\text{Yep} = \sum_{n=0} \frac{x^{2n+1}}{(2n+1)!}$$

④ $\cosh x = \dfrac{e^{x} - e^{-x}}{2} = \displaystyle\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$

⑤ $f(x) = \sin x$

$$f(0) + \frac{f'(0)\,x}{1!} + \frac{f''(0).\,x^2}{2!} + \frac{f'''(0).\,(x)^3}{3!} + \frac{f''''(0).\,(x)^4}{4!} + \frac{f^{(5)}(0).\,(x)^5}{5!} + \cdots$$

$$0 + \frac{1\,x}{1!} + 0 - \frac{1.x^3}{3!} + \frac{1.x^5}{5!} + 0 - \frac{x^7}{7!} + \cdots$$

$$= \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n . \frac{x^{2n+1}}{(2n+1)!}$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n . \frac{x^{2n}}{(2n!)}$$

$f(x) = \sin x$
$f(0) = \sin 0 = 0$

$f' = \cos x$
$f'(0) = \cos 0 = 1$

$f'' = -\sin x$
$f''(0) = 0$

$f^{(3)} = -\cos x$
$f^{(3)}(0) = -1$

$f^{(4)} = \sin x \longrightarrow f^{(4)}(0) = 0$
$f^{(5)} = \cos x \longrightarrow f^{(5)}$
$\longrightarrow f^{(6)} = 0 \quad f^{(7)} = -1$

Q.: Is the following conv. or div. :-

$$\sum_{1}^{\infty} \left(\frac{1}{2}\right)^n + \frac{\ln n}{n^2}$$

$$\sum_{1}^{\infty} \left(\frac{1}{2}\right)^n + \sum \frac{\ln n}{n^2}$$

geom + using L.C.T ~~geom~~

$|r| = \frac{1}{2} < 1$

$b_n = \frac{1}{n^{1.5}}$ Conv.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\ln n}{n^2} * \frac{n^{1.5}}{1}$$

$$\lim_{n\to\infty} \frac{\ln n}{n^{.5}} \quad \frac{\infty}{\infty} \;!! \; L'rule$$

$$\lim \frac{\frac{1}{n}}{0.5\, n^{-0.5}} \quad \lim_{n\to\infty} \frac{1}{n^{\frac{1}{2}}}$$

$$= \frac{1}{\infty} = \underline{0}$$

$$\Longrightarrow \sum \frac{\ln n}{n^2} \quad \text{conv.}$$

conv. + conv. $\longrightarrow$ conv.

دائماً $0! = 1$

⑥ $\sin^2 x = f(x)$

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

$$\frac{1}{2} - \frac{1}{2}\cos 2x$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n}}{(2n)!}$$

$$\cos 2x = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot (2x)^{2n}}{(2n)!}$$

$$\sin^2 x = \frac{1}{2} - \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n \cdot (2x)^{2n}}{(2n)!}$$

ال cos زي ال sin بس بدالها +

$$\cos^2 x = \frac{1}{2} + \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n \cdot (2x)^{2n}}{(2n)!}$$

⑦ $f(x) = \dfrac{1}{1-x}$

$f(x) = (1-x)^{-1} \longrightarrow f(0) = 1 = 0!$

$f'(x) = -1(1-x)^{-2} \times -1 \longrightarrow f'(0) = 1 = 1!$

$f''(x) = 2.1\,(1-x)^{-3} \longrightarrow f''(0) = 2.1 = 2!$

$f'''(x) = 3.2.1\,(1-x)^{-4} \longrightarrow f'''(0) = 3.2.1 = 3!$

$f^{(4)}(x) = 4.3.2.1\,(1-x)^{-5} \longrightarrow$

$\cancel{f''''(x) =}$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)\, x^n}{n!} = f(0) + \frac{f'(0)x}{1!} + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \cdots$$

يتبع

$$= 1 + \frac{1!\cdot x}{1!} + \frac{2!\ x^2}{2!} + \frac{3!\cdot x^3}{3!} + \cdots$$

$$= 1 + x + x^2 + x^3 + x^4 + \cdots$$

$$\Rightarrow \boxed{\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n} \quad \underline{\underline{\text{Exp}}}$$

$$= (1-x)^{-1}$$

(8) $\frac{1}{1+x}$. $f(x)$

$$\frac{1}{1-x} = \sum_{n=0} x^n$$

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n \cdot x^n$$

(9) $f(x) = \frac{x}{1+x^2}$

$$= x * \frac{1}{1-(-x^2)}$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n ,$$

$$\frac{x}{1+x^2} = x * \sum_{n=0} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n \cdot x^{2n+1}$$

(10) $\cancel{f(x) = \frac{1}{2+x}}$ (10) $f(x) = \dfrac{1}{2+x}$

$$= \frac{1}{2\left(1+\frac{x}{2}\right)} = \frac{1}{2}\left(\frac{1}{1-\left(\frac{-x}{2}\right)}\right)$$

$$= \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\frac{1}{2+x} = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{-x}{2}\right)^n = \sum_{n=0} \frac{(-1)^n \cdot x^n}{2^{n+1}}$$

(11) $(1+x)^K = f(x)$

$$f(x) = (1+x)^K \longrightarrow f(0) = 1$$
$$f'(x) = K\cdot(1+x)^{K-1} \longrightarrow f'(0) = K$$
$$f''(x) = K\cdot(K-1)(1+x)^{K-2} \longrightarrow f''(0) = K(K-1)$$
$$f'''(x) = K\cdot(K-1)(K-2)(1+x)^{K-3} \longrightarrow f'''(0) = K(K-1)(K-2)$$

$$f^{(n)}(\cancel{x}) = K(K-1)(K-2)\cdots\underbrace{(K-(n-1))}_{(K-n+1)}(1+x)^{K-n} \longrightarrow$$

$$\boxed{f^{(n)}(0) = K(K-1)(K-2)\cdots(K-n+1)}$$

$$f(x)\ \sum_{n=0}^{\infty}\frac{f^{(n)}(0)\,x^n}{n!} = \sum_{n=0}^{\infty}\frac{K(K-1)(K-2)\cdots(K-n+1)}{n!}\cdot x^n$$

$$= (1+x)^K$$

$\forall K$

نظرية ذات الحدين ← Binomial Series

(12) $f(x) = \dfrac{1}{\sqrt{1-x}}$

$= (1-x)^{-\frac{1}{2}}$ $\qquad -x \leftarrow x$ بدل

$= \left(1+(-x)\right)^{-\frac{1}{2}}$ $\qquad \frac{1}{2} \leftarrow K$ بدل

$$\sum_{n=0}^{\infty} \frac{\left(-\frac{1}{2}\right)\left(\frac{-1}{2}-1\right)\left(\frac{-1}{2}-2\right)\cdots\left(\frac{-1}{2}-n+1\right)(-x)^n}{n!}$$

(13) $f(x) = \dfrac{1}{(1+x^2)^3} = (1+x^2)^{-3}$

$x^2 \leftarrow x$ بدل

$-3 \leftarrow K$ بدل

* عند إيجاد ال sum ل Series ننظر أولاً إلى المضروب (factorial)
إذا كان فيها in factorial نفكر ب e^x أو e^{-x} في حالة $(2n)!$
نفكر بال cos و cosh في حالة $(2n+1)!$ نفكر بال sin أو بال sinh

*Ex:- Find the sum..

① $\frac{2}{1!} + \frac{4}{2!} + \frac{8}{3!} + \frac{16}{4!} + \cdots$

$2^1 \quad 2^2 \quad 2^3 \quad 2^4$

$\sum_{n=0} \frac{2^n}{n!}$ $\qquad e^x = \sum_{n=0} \frac{x^n}{n!}$

$= \sum_{n=0}^{\infty} \frac{2^n}{n!} - (n=0)$

$= e^2 - \left(\frac{2^0}{0!}\right) = e^2 - 1$

$\boxed{2}$ $1 - \ln 3 + \frac{(\ln 3)^2}{2!} - \frac{(\ln 3)^3}{3!} + \frac{(\ln 3)^4}{4!} - \cdots$

alternating

$= \sum_{n=0} (-1)^n \cdot \frac{(\ln 3)^n}{n!}$ $\qquad n!/(-1)^n \Downarrow$ $\quad e^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^n}{n!}$

///

$e^{-\ln 3} = e^{\ln 3^{-1}} = \frac{1}{3}$

نتظر الـ -1^n في حالة وجوده نفكر e^{-x} و $\sin x$ و $\cos x$.

[3] $$\pi - \frac{\pi^3}{3!} + \frac{\pi^5}{5!} - \frac{\pi^7}{7!} + \cdots$$

$$= \sum_{n=0} (-1)^n \cdot \frac{\pi^{2n+1}}{(2n+1)!} \longrightarrow \sin x = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n+1}}{(2n+1)!}$$

///

$\sin \pi = 0$

* Ex: Find the Taylor series for

(1) $f(x) = e^x$ at $a = 3$

$$e^x = e^{x-3+3} = e^{(x-3)} \cdot e^3$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$e^{x-3} = \sum_{n=0}^{\infty} \frac{(x-3)^n}{n!}$$

$$\boxed{e^x = e^3 \cdot \sum_{n=0}^{\infty} \frac{(x-3)^n}{n!}}$$

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② $f(x) = e^x$ at $a = -3$

$$e^x = e^{x+3-3} = e^{-3} \cdot e^{(x+3)}$$

$x+3$

$$= e^{-3} \cdot \sum_{n=0}^{\infty} \frac{(x+3)^n}{n!}$$

③ $f(x) = \sin x$ at $a = \frac{\pi}{2}$

$$\sin x = \sin\left(\overset{a}{\left(x - \frac{\pi}{2}\right)} + \overset{b}{\frac{\pi}{2}}\right)$$

$x - \frac{\pi}{2}$

$$\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b$$

$$= \cancel{\sin\left(x - \frac{\pi}{2}\right) \cdot \cos\frac{\pi}{2}}^{\,0} + \cos\left(x - \frac{\pi}{2}\right) \sin\frac{\pi}{2}$$

1

$$= \cos\left(x - \frac{\pi}{2}\right) \quad \text{but } \cos x = \sum_{n=0} (-1)^n \cdot \frac{x^{2n}}{(2n)!}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{\left(x - \frac{\pi}{2}\right)^{2n}}{(2n)!}$$

* Rule **

① $e^x = \sum_{n=0} \frac{x^n}{n!}$

② $e^{-x} = \sum_{n=0} (-1)^n \frac{x^n}{n!}$

③ $\sin x = \sum_{n=0} (-1)^n \cdot \frac{x^{2n+1}}{(2n+1)!}$

④ $\cos x = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n}}{(2n)!}$

⑤ $\sinh = \sum_{n=0} \frac{x^{2n+1}}{2n+1!}$

⑥ $\cosh = \sum$

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* **H.w**

$$f(x) = \sin x \quad at \quad a = \frac{\pi}{3}$$

* identity

$$\cos(a \pm b) = \cos a . \cos b \pm \sin a . \sin b$$